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Abstract

In this work we construct two representations, S_G and St_G, for the general linear group $G = GL_n(\mathbb{Z}/p^h\mathbb{Z})$, over the integers modulo a prime power, which, in different ways may be regarded as analogues of the usual Steinberg representation S_{G_1} of $G_1 = GL_n(\mathbb{Z}/p\mathbb{Z})$.

Chapter 1 contains technical results which are required for the representation theory of the sequel. We investigate the parabolic structure of G and discuss 'unramified semisimple' and 'regular' elements of G.

In Chapter 2 we construct S_G by an analogue of Curtis' formula; this is inductive on h, requiring prior construction of S_M for $M \cong GL_n(\mathbb{Z}/p^{h-1}\mathbb{Z})$. We also express S_G as an alternating sum of permutation representations. S_G is not irreducible if $h \geq 2$ but its character is 0 or $\pm$ a power of p at many elements.

In Chapter 3 St_G appears as the 'largest' irreducible component of 1_B^G (suitably defined) and is constructible homologically, but it has complicated character values. It is contained in the 'Gelfand-Graev representation' χ_U^G with multiplicity 1; this is proved by considering a related 'affine Steinberg representation' St_H, which is isomorphic to χ_U^H. We also show that χ_U^G is multiplicity-free in certain cases.

In Chapter 4 we give geometric interpretations of S_G and St_G using a Bruhat-Tits building; these enable us to show that S_G is a subrepresentation of 1_B^G and contains St_G (except possibly if p=2).

Finally, in Chapter 5 we give some examples and counter-examples; in particular we show that the character of S_G is not always 0 or $\pm$ a power of p and we compute the character of St_G at split semisimple elements for n=2 and 3, giving a conjecture for the general case.

Introduction

The purpose of this work is to construct representations for the general linear groups $G = GL_n(R)$, over the ring $R = \mathbb{Z}/p^h\mathbb{Z}$ of integers modulo a prime power, which may be regarded as analogues of the usual Steinberg representation S_{G_1} of $G_1 = GL_n(\mathbb{Z}/p\mathbb{Z})$. More generally, we may take R to be a proper quotient of the maximal compact subring of a non-archimedean local field K - such a ring is finite and has the same essential properties as $\mathbb{Z}/p^h\mathbb{Z}$.

In fact we shall give two representations which, in different ways, play the role of 'Steinberg representation' of G. The first one, S_G, has character values which are zero or $\pm$ a power of p at many elements of G; it coincides with S_{G_1} if h=1 but it is not irreducible if $h \geq 2$, although it is still a subrepresentation of 1_B^G, suitably defined (except possibly if p=2). The second representation, St_G, is the 'largest' irreducible component of 1_B^G in the sense that its degree, as a polynomial in p, has the same leading term as the degree of 1_B^G; thus it coincides with S_{G_1} in case h=1, and in fact is contained in S_G in case $h \geq 2$ (except possibly if p=2). However, in case $h \geq 2$ its character values are quite complicated. St_G is constructible homologically; the constructions in the two cases h=1 and $h \geq 2$ are somewhat different, though related. St_G is probably also the unique common irreducible component of 1_B^G and the Gelfand-Graev representation χ_U^G (this is true if n=2 or 3). Both S_G and St_G may be expressed as alternating sums of permutation representations.

The work is organised as follows. The first chapter consists mainly of technical results necessary for the sequel. We regard

G as the automorphism group of a free R-module V of rank n and with this point of view we define a parabolic subgroup P of G to be the stabiliser of a flag of free submodules of V; we prove that P is self-normalising in G and we give a 'Levi decomposition' of P. We also study 'root subgroups' of G. Our methods of proof are elementary in contrast to those of [6].

Next we give some 'distinguished representatives' for the cosets of P in G, making use of distinguished representatives for the cosets of a parabolic subgroup W_P of the Weyl group W of G_1 (cf. [5]). In this connection we also give a combinatorial lemma to deal with alternating sum expressions occuring in Chapters 2 and 4. Chapter 1 ends with a discussion of semi-simple and regular elements of G; we restrict our attention to 'unramified' semisimple elements, which are those which split over an extension R' of R corresponding to an unramified extension K' of K. We derive a criterion for such elements to be regular and we also prove that G has a unique regular unipotent conjugacy class.

In Chapter 2 we define S_G as an alternating sum of representations induced from parabolic subgroups P of G analogous to Curtis' formula for S_{G_1} ([5]); here we induce, not the unit representation in general, but the lift to P of the representation $S_{L(h-1)}$ of a Levi subgroup L of P over $\mathbb{Z}/p^{h-1}\mathbb{Z}$ (this is 1 if h=1). We show that S_G may also be expressed as the alternating sum of permutation representations $1_{H_{\mathbf{k}}}^{G}$ parametrised by (n-1)-tuples of integers $\mathbf{k} = (k_1, \ldots, k_{n-1})$ with $0 \leq k_r \leq h$ for each r (or alternatively by filtrations of the set of simple roots); the $H_{\mathbf{k}}$ are precisely the subgroups of G for which one constructs principal series representations $\lambda_{H_{\mathbf{k}}}^{G}$ (λ a suitable character of a split torus of G) ([10]), thus providing a second analogy with Curtis' formula. We show that the character of S_G at unramified semisimple elements t is $\pm$ a power of p, the sign being related to the number of orbits of the Frobenius map on the set of eigenvalues of t. We also show that the character of S_G vanishes at non-trivial unipotent elements 'near the identity'. In case h=1 our computations cover all the conjugacy classes of G_1. The method of computation, which is illustrated by the case h=1 and t lying in a minimal parabolic subgroup (e.g. t split semisimple), is first to show that the set of parabolic subgroups of G of a given type and the set of minimal parabolic subgroups of G 'lying over' the distinguished representatives of a suitable parabolic subgroup of W have the same number of elements fixed by t. Then we use the alternating sum formula for S_G to show that $S_G(t)$ equals the number of minimal parabolic subgroups fixed by t and lying over w_0 (the longest element of W), i.e. 'opposite' to a fixed such group. The idea of this proof is related to Tits' computation of the homology of the spherical building associated to G_1 ([17]).

In Chapter 3 we construct St_G for $h \geq 2$ as the homology of a simplicial complex with simplices corresponding to certain flags of submodules of V which here play a role analogous to flags of subspaces of V in case h=1 (the analogy will be made clearer in Chapter 4). In fact this method yields a whole family of irreducible components $R(\underset{\sim}{a})$ of 1_B^G of which St_G is the largest. We prove that the character of $R(\underset{\sim}{a})$ vanishes on 'regular enough' unramified semisimple elements (unlike the case h=1). Related to St_G is an irreducible representation St_H of the affine group which is constructed in a similar manner to St_G; The 'geometric' construction shows that St_H is analogy with Curtis' formula. We show that the character of S_G at unramified semisimple elements t is $\pm$ a power of p, the sign being related to the number of orbits of the Frobenius map on the set of eigenvalues of t. We also show that the character of S_G vanishes at non-trivial unipotent elements 'near the identity'. In case h=1 our computations cover all the conjugacy classes of G_1. The method of computation, which is illustrated by the case h=1 and t lying in a minimal parabolic subgroup (e.g. t split semisimple), is first to show that the set of parabolic subgroups of G of a given type and the set of minimal parabolic subgroups of G 'lying over' the distinguished representatives of a suitable parabolic subgroup of W have the same number of elements fixed by t. Then we use the alternating sum formula for S_G to show that $S_G(t)$ equals the number of minimal parabolic subgroups fixed by t and lying over w_0 (the longest element of W), i.e. 'opposite' to a fixed such group. The idea of this proof is related to Tits' computation of the homology of the spherical building associated to G_1 ([17]).

In Chapter 3 we construct St_G for $h \geq 2$ as the homology of a simplicial complex with simplices corresponding to certain flags of submodules of V which here play a role analogous to flags of subspaces of V in case h=1 (the analogy will be made clearer in Chapter 4). In fact this method yields a whole family of irreducible components $R(\underset{\sim}{a})$ of 1_B^G of which St_G is the largest. We prove that the character of $R(\underset{\sim}{a})$ vanishes on 'regular enough' unramified semisimple elements (unlike the case h=1). Related to St_G is an irreducible representation St_H of the affine group which is constructed in a similar manner to St_G; The 'geometric' construction shows that St_H is contained in the restriction of St_G to H, but to prove St_H is irreducible we employ an algebraic method, showing first that $St_H \cong \chi_U^H$ for any non-singular character χ of the unipotent subgroup U of G. This also yields the fact that St_G is contained in the 'Gelfand-Graev representation' χ_U^G with multiplicity 1. In view of [7] it is natural to ask whether χ_U^G is multiplicity-free, and we are able to show that this is the case if n=2 or 3, pointing out the difficulties in the general case. We also show that the number of components of χ_U^G in case n=3 and $h \geq 2$ is greater than the number of regular conjugacy classes of G, unlike the case h=1, when there is equality. We conclude Chapter 3 by pointing out how St_G occurs as part of the principal series of representations of G.

In Chapter 4 we give geometric interpretations of S_G and St_G by describing the representations $1_{H_k}^G$ as permutation representations on certain sets of subcomplexes of (a finite subcomplex X of) the Bruhat-Tits building of $GL_n(K)$. This enables us to relate the construction of St_G ($h \geq 2$) to that of St_{G_1} and also to prove that S_G is a subrepresentation of 1_B^G and contains St_G (except possibly if p=2). We show how to construct the Bruhat-Tits building using (equivalence classes of) lattices in a K-vector space of dimension n; it follows that the vertices of X correspond to the R-submodules of V of length $\leq n-1$. We briefly consider how such geometric constructions of S_G and St_G may be made for more general groups.

Finally, in Chapter 5 we give some examples and counterexamples. We consider the problem of decomposing 1_B^G : recall that in case h=1 the B-orbits on 1_B^G (which is the permutation representation of G on the set of complete flags in V) have a simple geometric characterisation and are parametrised by the elements of the Weyl group $W \cong S_n$. The distinct irreducible components of 1_B^G are parametrised by the conjugacy classes of W as are also the conjugacy classes of unipotent elements of G (though not canonically). However, in case n=3 and $h \geq 3$ we show that the corresponding simple 'geometry' no longer characterises the B-orbits on 1_B^G and moreover both their number and the number of unipotent conjugacy classes in G are not bounded independently of p. In cases n=2 and n=3,h=2 we decompose 1_B^G and S_G completely; in these cases the number of distinct irreducible components of 1_B^G equals the number of unipotent conjugacy classes in G. Next we give an example (when n=3,h=3) to show that the character of S_G is not always 0 or $\pm$ a power of p. In the final section we compute the character of St_G at all split semisimple elements of G in the cases n=2 and n=3; these results lead to a conjecture for the general case.

Accounts of the Steinberg representation in case h=1 (when R is a field) can be found in [5] and [17]. In case $h \geq 2$ the representation St_G was well-known for GL_2 (see for example [16]) and G.Lusztig suggested to me how one might construct it for GL_n. G.Lusztig also established the existence of the representation S_G with 'nice' character values for GL_2 and suggested that such a representation might exist for GL_n. I wish to thank him for explaining to me the case h=1 and for suggesting what to look for in case $h \geq 2$.

<u>Notations and conventions</u>

All representations will be assumed to be complex unless otherwise specified.

$\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{C}$ the field of complex numbers.

If M is a module over a ring R and $e_1,\ldots,e_m \in M$ then

$$< e_1,\ldots,e_m > = M \quad \text{or} \quad < e_i : 1 \leqslant i \leqslant m > = M$$

means that the set $\{e_1,\ldots,e_m\}$ generates M as an R-module.

If G is a group and $g,h \in G$ then ${}^{g}h$ means ghg^{-1}; and if H is a subgroup of G then ${}^{g}H$ means gHg^{-1}.
Also $N_G(H) = \{g \in G : {}^{g}H = H\}$ and $Z_G(h) = \{ g \in G : {}^{g}h = h\}$.
The commutator $[g,h]$ means $g^{-1}h^{-1}gh$.

■ indicates the end of a proof.

Chapter 1

The parabolic structure of the general linear group.

This chapter is mainly concerned with those group-theoretic results required for the representation theory of chapters 2 and 3. First we develop the theory of parabolic, Levi and root subgroups of the general linear group $G = GL_n(R)$ over a certain finite local ring R (see §1.0) from a 'geometric' point of view using flags of R-modules. In case R is a finite field the results are well-known ([1], [2]). Some of our results are contained in those of [6] and [9] but our methods are more elementary. We also determine some 'distinguished' representatives for cosets of a parabolic subgroup of G and compute its order; finally we discuss regular and semi-simple elements of G.

1.0 The ring R

Let K be a non-archimedean local field with valuation v. Its maximal compact subring $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ is a local ring with maximal ideal $\mathfrak{p} = \{x \in K \mid v(x) > 0\}$. In fact $\mathcal{O}$ is a complete discrete valuation ring so its ideals are precisely $\mathfrak{p}^h$ $(0 \leq h \in \mathbb{Z})$ and $\mathfrak{p}$ is the only non-zero prime ideal: the residue field $k = \mathcal{O}/\mathfrak{p}$ is finite, of characteristic p say. Let $|k| = q$. Two cases arise:

Case (i) : $\mathrm{char}(\mathcal{O}) = p$.

Then $\mathcal{O} \simeq k[[X]]$, a formal power series ring, and $K \simeq k((X))$.

Case (ii) : $\mathrm{char}(\mathcal{O}) = 0$.

Then $\mathcal{O}$ is the completion of the ring of integers $\mathcal{O}_F$ in a number field F at some prime ideal P of $\mathcal{O}_F$, and K is the completion of F for the P-adic valuation on F; moreover $\mathcal{O}$ is the closure of $\mathcal{O}_F$ in K.

We refer to [15] or [19] for the theory of local fields.

Definition

Let h be a positive integer ≥ 1. Define $R = \mathcal{O}/\mathfrak{p}^h$. (This notation will be standard in the sequel except that sometimes we shall point out results which hold for more general rings R).

Then R is a finite local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}/\mathfrak{p}^h$. The ideals of R are $\mathfrak{m}^i = \mathfrak{p}^i/\mathfrak{p}^h$ $(0 \leq i \leq h)$. If h=1 then R is a field ($R \cong k$) but if $h > 1$ then R is principal but not an integral domain (e.g. if $0 \neq x \in \mathfrak{m}^{h-1}$ then $x^2 = 0$).

There is a unique map $\sim : k \to \mathcal{O}$ (the 'Teichmuller section' in case (ii)) such that the composition $k \xrightarrow{\sim} \mathcal{O} \to \mathcal{O}/\mathfrak{p} = k$ is the identity map and such that $\widetilde{\lambda^p} = \tilde{\lambda}^p$ and $\widetilde{\lambda\mu} = \tilde{\lambda}\tilde{\mu}$ $(\lambda, \mu \in k)$. Let ξ be an irreducible element of $\mathcal{O}$ (it is unique up to multiplication by a unit and generates $\mathfrak{p}$). Then each element of $\mathcal{O}$ has a unique expression in the form $\sum_{i=0}^{\infty} \tilde{a}_i \xi^i$ with $a_i \in k$. It is clear how R inherits these properties. In particular if π generates $\mathfrak{m}$ then each element of R has a unique expression in the form $r = \sum_{i=0}^{h-1} \tilde{a}_i \pi^i$ $(a_i \in k,\ 0 \leq i \leq h-1)$.

We shall write $r \| \pi^m$ iff $a_i = 0$ $(0 \leq i \leq m-1)$ and $a_m \neq 0$, i.e. $r \in \mathfrak{m}^m - \mathfrak{m}^{m-1}$. Write $r \| s$ if $r \| \pi^m$ and $s \| \pi^l$ imply that m=l; note that $r \| s$ iff $r | s$ and $s | r$.

We shall also need the following notations: for $1 \leq i \leq h$ define $R^{(i)} = 1 + \mathfrak{m}^i = \{1 + x\pi^i : x \in R\}$; then $R^{(i)} = \ker(\pi_i)$ where $\pi_i : R \to R_i = R/\mathfrak{m}^i$ is 'reduction mod π^i'. We may also write $\pi_1(r) = \bar{r}$ for $r \in R$ and $\pi_1(R) = \bar{R}$ $(\cong k)$. $R^{(0)} = R^*$ will denote the group of units in R and R^+ the additive group of R. Note that $|R| = q^h$, $|R^{(i)}| = |\mathfrak{m}^i| = q^{h-i}$ $(1 \leq i \leq h)$, and $|R^*| = q^{h-1}(q-1)$.

1.01 Unramified extensions of R

Let K' be a finite algebraic extension of K with valuation v'. Then the maximal compact subring $\mathcal{O}'$ is the integral closure of $\mathcal{O}$ in K' and the extended ideal $\mathfrak{p}' = \mathcal{O}'.\mathfrak{p}$ is a power of the non-zero prime ideal of $\mathcal{O}'$. The extension is unramified iff $\mathfrak{p}'$ is prime. In this case ξ is also irreducible in $\mathcal{O}'$. Equivalently the extension is unramified iff $v'(\xi) = 1$.

Definition

An unramified extension R' of R is a ring of the form $R' = \mathcal{O}'/\mathfrak{p}'^h$ where (in the above notation) K' is an unramified extension of K.

In case (ii), writing $v(p) = e$, p is irreducible in $\mathcal{O}$ iff $e = 1$, in which case K is an unramified extension of the p-adic numbers $\mathbb{Q}_p$ and $\mathcal{O} \cong W(k)$, the ring of Witt vectors over k ([15], Chap. II). If $|k| = p$ then $\mathcal{O} \cong \mathbb{Z}_p$, the ring of p-adic integers, and $R \cong \mathbb{Z}/p^h\mathbb{Z}$

1.02 The Frobenius automorphism

Up to isomorphism there is a unique unramified extension K' of K of each degree f and a corresponding unique R'; such extensions correspond bijectively to the finite field extensions k' of k. The Galois group $Gal(K'/K) \simeq Gal(k'/k)$ is cyclic, generated by the Frobenius automorphism F which induces the permutation $\mu \mapsto \mu^q$ on the group of roots of 1 of order prime to p (which generate K' over K) (see [19], § 1.4). So we have a Frobenius automorphism $F:R' \to R'$ given by $\sum_{i=0}^{h-1} a_i\pi^i \mapsto \sum_{i=0}^{h-1} a_i^q\pi^i$, and $R = R'^F = \{r \in R' \mid F(r) = r\}$, since $k = k'^F$.

1.03 Modules over R

Let M be a finitely generated R-module. Then M may be regarded as a module over the principal ideal domain $\mathcal{O}$, with $\mathfrak{p}^h.M = 0$, and the structure of such modules is known to be as follows. (see also [19], §2.2).

Proposition 1.04

(i) There exist unique integers $a_1, a_2, \dots, a_n$ with $h \geq a_1 \geq \dots \geq a_n > 0$ such that there is a decomposition $M = M_1 \oplus \dots \oplus M_n$ with $M_i \cong R/m^{a_i}$ as R-modules.

Also $|M| = q^a$ where $a = \sum_{i=1}^{n} a_i$.

(ii) If M is a free R-module (so $a_i = h$ for $1 \leq i \leq n$) and if N is a free R-submodule of M then there exists a free R-submodule N' of M such that $M = N \oplus N'$; in particular, if $\mathrm{rank}(N) = \mathrm{rank}(M)$ then $M = N$.

The sequence $(a_1, \dots, a_n)$ will be called the <u>type</u> of the module M. Also write $M \sim (a_1, \dots, a_n)$. $n = l(M)$ is called the <u>length</u> of M.

We remark that if $b_i = h - a_i$ and $c_i = l(m^{i-1}M/m^iM)$ for $1 \leq i \leq h$ then $(c_1, \dots, c_h)$ is the partition conjugate to $(b_1, \dots, b_n)$, i.e. $c_i = |\{j : b_j \geq i\}|$.

1.1 Flags and parabolic subgroups

Let V be a free R-module of finite rank n and let $G = GL(V)$ be the set of R-module isomorphisms $g: V \xrightarrow{\cong} V$.

Definition

Let $X = \{i_1, \dots, i_r\}$ be a subset of $S = \{1, 2, \dots, n-1\}$ with $i_1 < \dots < i_r$ (this notation will be standard henceforth). Then a <u>flag of type X</u> in V is a flag $\sigma = (0 \subset U_{i_1} \subset \dots \subset U_{i_r} \subset V)$ of R-submodules of V with U_{i_k} free of rank i_k $(1 \leq k \leq r)$.

Let $\mathcal{F}_X = \{$flags of type X in V$\}$. Note that G acts on $\mathcal{F}_X$ by

$$g.\sigma = (0 \subset g.U_{i_1} \subset \dots \subset g.U_{i_r} \subset V) \qquad (g \in G).$$

Now fix a <u>standard flag</u> $\rho_S = (0 \subset V_1 \subset \dots \subset V_{n-1} \subset V) \in \mathcal{F}_S$ and let $B = P_S = \mathrm{Stab}_G(\rho_S)$, the stabiliser of ρ_S for the action of G (a subgroup of G). Then for each $X \subset S$ we also have the <u>standard flag of type X</u> : $\rho_X = (0 \subset V_{i_1} \subset \dots \subset V_{i_r} \subset V)$,and

the <u>standard parabolic subgroup of type X</u> : $P_X = \mathrm{Stab}_G(\rho_X)$.
Note that $P_\emptyset = G$.
We also fix a free basis $\underset{\sim}{e} = \{e_1, \dots, e_n\}$ of V such that for $1 \leqslant i \leqslant n-1$ we have $V_i = \langle e_1, \dots, e_i \rangle$.

<u>Lemma 1.11</u>

There is a G-equivariant bijection $\theta : G/P_X \to \mathfrak{F}_X$ given by $gP_X \mapsto g.\rho_X$ $(g \in G)$.

In particular, any $\sigma \in \mathfrak{F}_X$ has the form $\sigma = g.\rho_X$ where $g \in G$ is 'unique modulo P_X'.

(Here G acts on G/P_X by translation : $g.(hP_X) = (gh)P_X$.)

<u>Proof</u>:

Let $g_1, g_2 \in G$. Then $g_1P_X = g_2P_X$ iff $g_2^{-1}g_1 \in P_X$ iff $g_2^{-1}g_1.\rho_X = \rho_X$ iff $g_1.\rho_X = g_2.\rho_X$: so θ is well-defined and injective. Moreover $\theta(g_1.(g_2P_X)) = \theta((g_1g_2)P_X) = (g_1g_2)\rho_X$ $= g_1.(g_2.\rho_X) = g_1.\theta(g_2P_X)$ so θ is G-equivariant.

To see θ is surjective we show that G acts transitively on $\mathfrak{F}_X$. So let $\sigma = (0 \subset U_{i_1} \subset \dots \subset U_{i_r} \subset V) \in \mathfrak{F}_X$ and choose a free basis $\{f_1, \dots, f_n\}$ of V such that $U_{i_k} = \langle f_1, \dots, f_{i_k} \rangle$ $(1 \leqslant k \leqslant r)$. Then the map $g : V \to V$ sending e_i to f_i $(1 \leqslant i \leqslant n)$ is in G and $\sigma = g.\rho_X$. So if $\sigma' \in \mathfrak{F}_X$ then there exists $g' \in G$ with $\sigma' = g'.\rho_X$ and $\sigma' = g'g^{-1}.\sigma$. ∎

<u>Lemma 1.12</u>

(i) $\mathrm{Stab}_G(g.\rho_X) = {}^gP_X$ $(g \in G)$.
(ii) If $g \in G$ and ${}^gP_X = P_X$ then $g \in P_X$. (i.e. $N_G(P_X) = P_X$).

<u>Proof</u>:

$$
\begin{aligned}
\text{(i) } \mathrm{Stab}_G(g.\rho_X) &= \{h \in G \mid hg.\rho_X = g.\rho_X\} \\
&= \{gkg^{-1} \mid k \in G \text{ and } k.\rho_X = \rho_X\} \\
&= gP_Xg^{-1} = {}^gP_X.
\end{aligned}
$$

(ii) Suppose $g \notin P_X$. Then there exists $i_j \in X$ such that $g.V_{i_j} \neq V_{i_j}$. Take the least such i_j and let $U = V_{i_j}$. Then we shall show that there exists $p \in P_X$ such that $p.U \neq U$, whence ${}^gP_X \neq P_X$ (using (i)). This proves (ii).

Now with respect to the basis $\underset{\sim}{e}$ of V, g may be written in block matrix form (A_{lm}) where for $1 \leq l,m \leq r+1$ A_{lm} is an $(i_l - i_{l-1}) \times (i_m - i_{m-1})$ matrix, $A_{lm} = 0$ if $m < l$ and $m < j$, and $A_{kj} \neq 0$ for some k with $k > j$. Now there exist block diagonal matrices $p_1 = (S_l)$, $p_2 = (T_l) \in P_X$, with $S_l = 1_{i_l - i_{l-1}}$ for $l \neq k$, and $T_l = 1_{i_l - i_{l-1}}$ for $l \neq j$, and so that $A'_{kj} = S_k A_{kj} T_j$ has entries (a'_{pq}) which are zero if $p \neq q$, and at least one with $p = q$, say a'_{11}, is non-zero. Since $p_1, p_2 \in P_X$ we may replace g by $p_1 g p_2$ and A_{kj} by A'_{kj}.

Now since $V_{i_j} \neq g.V_{i_j}$ there exists at least one i such that $i_{j-1} < i \leq i_j$ and $e_i \notin g.V_{i_j}$; thus we may choose k, S_k and T_j so that in fact $a'_{11} \in R^*$. Now let $p = (E_{lm}) \in P_X$ be a block matrix with $E_{ll} = 1_{i_l - i_{l-1}}$, $E_{lm} = 0$ for $l \neq m$, except for E_{jk}, which may be chosen so that $D = E_{jk} A'_{kj}$ has zero entries except for a 1 in the $(i - i_{j-1}, i - i_{j-1})$ - place. Then D acts on the module $< e_{i_{j-1}+1}, \ldots, e_{i_j} >$ and $D.e_l = \delta_{il} e_i$ $(i_{j-1} < l \leq i_j)$. Further, $pg.e_l = g.e_l$ for $1 \leq l \leq i_j$, $l \neq i$, but $pg.e_i = g.e_i + e_i$, which doesn't lie in $g.V_{i_j} = U$. Hence $p.U \neq U$. ∎

<u>Definition</u>

A <u>parabolic subgroup</u> P of G <u>of type X</u> will mean a G-conjugate of P_X : $P = {}^gP_X$ for some $g = G$. Denote the set of such by $\mathcal{P}_X$. (<u>Warning</u>: this labelling of parabolic subgroups is the opposite to that employed by Bourbaki [1], where for example $G = P_S$, but is more convenient for us).

G acts on $\mathcal{P}_X$ by conjugation : $g.(P) = {}^gP$ $(g \in G,\ P \in \mathcal{P}_X)$. Also we may partially order the set $\mathcal{P} = \bigcup_{X \subseteq S} \mathcal{P}_X$ of all parabolic subgroups of G by inclusion.

Now define a partial order $\leqslant$ on $\mathfrak{F} = \bigcup_{X \subset S} \mathfrak{F}_X$ as follows.
Let $\sigma = (0 \subset U_{i_1} \subset \ldots \subset U_{i_r} \subset V) \in \mathfrak{F}_X$ and let
$\tau = (0 \subset W_{j_1} \subset \ldots \subset W_{j_s} \subset V) \in \mathfrak{F}_Y$. Then $\sigma \leqslant \tau$ iff (i) $X \subset Y$ and (ii) $U_i = W_i$ for $i \in X$.

The main result of this section is the following.

<u>Proposition 1.13</u>

For each $X \subset S$ there is a G-equivariant bijection $\varphi_X : \mathfrak{F}_X \to \mathcal{P}_X$ and these induce an order-reversing G-equivariant bijection $\varphi : \mathfrak{F} \to \mathcal{P}$ given by $\sigma \mapsto \mathrm{Stab}_G(\sigma)$.

<u>Proof</u>:

$\varphi_X : \sigma \mapsto \mathrm{Stab}_G(\sigma)$ is well-defined and surjective by Lemma 1.11 and Lemma 1.12(i) and is injective by Lemma 1.12(ii). And if $h \in G$, $\sigma = g.\rho_X \in \mathfrak{F}_X$ then $\varphi_X(h.\sigma) = \varphi_X(hg.\rho_X) = {}^{hg}P_X = {}^{h}({}^{g}P_X) = {}^{h}(\varphi_X(\sigma))$ so φ_X is G-equivariant.

Finally, to see that φ is order-reversing :
$\sigma \leqslant \tau \Rightarrow \mathrm{Stab}_G(\tau) \subset \mathrm{Stab}_G(\sigma)$ is clear from the definition of $\leqslant$. On the other hand if ${}^{g}P_Y \subset {}^{h}P_X$ then for each $i \in X$ ${}^{g}P_Y$ stabilises $g.V_i$, so P_Y stabilises $g^{-1}h.V_i$: so in fact P_S stabilises $g^{-1}h.V_i$ since $P_S \subset P_Y$. The proof of Lemma 1.12(ii) now shows $g^{-1}h.V_i = V_i$, so $h.V_i = g.V_i$, and in fact P_Y stabilises V_i $(i \in X)$, whence $X \subset Y$. Thus we have shown that $h.\rho_X \leqslant g.\rho_Y$. ■

<u>Corollary 1.14</u>

(i) If P is parabolic of type X, and $Y \subset X$ then there exists a unique parabolic Q of type Y such that $P \subset Q$.

(ii) If P, Q are parabolic of types X, Y respectively, and $P \subset Q$, then $Y \subset X$.

<u>Proof</u>: Immediate from Proposition 1.13. ■

Corollary 1.15

Let P,Q be parabolic with $P \cap Q$ also parabolic and ${}^gP \subset Q$ for some $g \in G$. Then $g \in Q$ and $P \subset Q$.

Proof:

Suppose $P,Q,P \cap Q$ have types X,Y,Z respectively. Then ${}^{g^{-1}}Q$ also has type Y and ${}^gP$ has type X. Now $P \cap Q \subset P$ and ${}^gP \subset Q$ so $Y \subset X \subset Z$ by Corollary 1.14(ii). So by Corollary 1.14(i) Q is the unique parabolic of type Y containing $P \cap Q$; but we also have ${}^{g^{-1}}Q \supset P \supset P \cap Q$ so ${}^{g^{-1}}Q = Q$, whence $g \in Q$ by Lemma 1.12(ii And $P \subset {}^{g^{-1}}Q = Q$, i.e. $P \subset Q$. ∎

Corollary 1.16

(i) Two parabolic subgroups of G with parabolic intersection are conjugate iff they are equal.

(ii) Every parabolic subgroup of G is equal to its own normaliser in G.

(iii) For $X \subset S$ and $g \in G$ $\quad {}^gP_X = \bigcap_{i \in X} {}^gP_{\{i\}}$. ∎

1.2 Decompositions and Levi components

For $X \subset S$ write $w_0(X) = \{n-i : i \in X\}$; so $S = w_0(S)$.

Definition

The flag $\sigma' = (0 \subset U'_{n-i_r} \subset \ldots U'_{n-i_1} \subset V) \in \mathcal{F}_{w_0(X)}$ is opposite to $\sigma = (0 \subset U_{i_1} \subset \ldots \subset U_{i_r} \subset V) \in \mathcal{F}_X$ iff for $1 \leqslant k \leqslant r$ we have $U_{i_k} \cap U'_{n-i_k} = 0$.

In this case we have $V = U_{i_k} \oplus U'_{n-i_k}$ for each k, by §1.04(ii

Assuming that σ' is opposite to σ we shall write $W_{i_k} = U_{i_k} \cap U'_{n-i_{k-1}}$ for $1 \leqslant k \leqslant r+1$, where $i_0=0, i_{r+1}=n$, and $U_0 = U'_0 = 0$. $U_n = U'_n = V$. (These and similar notational conventions will be understood in future).

Lemma 1.21

For $1\leq k\leq r+1$ W_{i_k} is free over R of rank (i_k-i_{k-1}) and we have the direct sum decomposition $V = W_{i_1} \oplus \ldots \oplus W_{i_{r+1}}$.

Proof:

We use induction on k. First note that $W_{i_1} = U_{i_1} \cap V = U_{i_1}$ which is free of rank $i_1 = (i_1-i_0)$ over R.

Assume now that for some integer k with $1\leq k\leq r$ we have :

(*) For $1\leq j\leq k$ W_{i_j} is free of rank (i_j-i_{j-1}) over R and $U_{i_j} = W_{i_1} \oplus \ldots \oplus W_{i_j}$.

$$\begin{aligned} \text{Now } W_{i_{k+1}} \cap U_{i_k} &= U_{i_{k+1}} \cap U'_{n-i_k} \cap U_{i_k} \\ &= U'_{n-i_k} \cap U_{i_k} \quad \text{since } U_{i_k} \subset U_{i_{k+1}} \\ &= 0, \end{aligned}$$

so the sum $U_{i_k} + W_{i_{k+1}}$ is direct; thus $U_{i_{k+1}} \supset U_{i_k} \oplus W_{i_{k+1}}$. (†)

Now consider the projection $pr:V = U_{i_k} \oplus U'_{n-i_k} \to U_{i_k}$ onto the first factor, which has kernel U'_{n-i_k}. Then

$\ker(pr|U_{i_{k+1}}) = \ker(pr) \cap U_{i_{k+1}} = U'_{n-i_k} \cap U_{i_{k+1}} = W_{i_{k+1}}$,

$$\begin{aligned} \text{so } U_{i_k} &= pr(U_{i_{k+1}}) \quad (\text{since } U_{i_{k+1}} \supset U_{i_k}) \\ &\cong U_{i_{k+1}}/\ker(pr|U_{i_{k+1}}) \\ &= U_{i_{k+1}}/W_{i_{k+1}} . \end{aligned}$$

Hence by Proposition 1.04 and (†) $W_{i_{k+1}}$ is free of rank $(i_{k+1}-i_k)$ over R and $U_{i_{k+1}} = U_{i_k} \oplus W_{i_{k+1}}$. So (*) holds with k+1 instead of k; the result now follows by induction. ■

Definition

A <u>decomposition</u> of σ is an (ordered) sequence $\Delta = (W_{i_1},\ldots,W_{i_{r+1}})$ of R-modules with W_{i_k} free of rank (i_k-i_{k-1}) and $U_{i_k} = W_{i_1} \oplus \ldots \oplus W_{i_k}$ $(1\leq k\leq r+1)$.

This last equation shows that $P = \mathrm{Stab}_G(\sigma)$ acts on such decompositions : $p.\Delta = (p.W_{i_1},\ldots,p.W_{i_{r+1}})$.

Lemma 1.22

There is a P-equivariant bijection

θ : {flags opposite σ} $\to$ {decompositions of σ} given by

$(0 \subset U'_{n-i_r} \subset \dots \subset U'_{n-i_1} \subset V) \to (W_{i_1}, \dots, W_{i_{r+1}})$,

where $W_{i_k} = U_{i_k} \cap U'_{n-i_{k-1}}$ $(1 \leq k \leq r+1)$.

Proof:

θ is well-defined by Lemma 1.21 and is bijective because $U'_{n-i_k} = W_{i_{k+1}} \oplus \dots \oplus W_{i_{r+1}}$ $(1 \leq k \leq r)$, which follows from the proof of Lemma 1.21.

Also if $p \in P$ then $\theta(p.\sigma') = (U_{i_1} \cap pU'_{n-i_0}, \dots, U_{i_{r+1}} \cap pU'_{n-i_r})$ $= (p.(U_{i_1} \cap U'_{n-i_0}), \dots, p.(U_{i_{r+1}} \cap U'_{n-i_r})) = p.\theta(\sigma')$, so θ is P-equivariant. ∎

Definition

A Levi component (or subgroup) of P is a group of the form $L = P \cap \mathrm{Stab}_G(\sigma')$ for some flag $\sigma' \in \mathcal{F}_{w_0(X)}$ opposite to σ. Keeping the preceding notation and also writing $\Delta = \theta(\sigma')$ we have :

Lemma 1.23

$L = \mathrm{Stab}_P(\Delta)$.

Proof:

L stabilises both U_{i_k} and $U'_{n-i_{k-1}}$ so stabilises W_{i_k} for each k. On the other hand, if $g \in P$ stabilises Δ , i.e. if g stabilises each $W_{i_k} = U_{i_k} \cap U'_{n-i_{k-1}}$ then the relations $U'_{n-i_k} = W_{i_{k+1}} \oplus \dots \oplus W_{i_{r+1}}$ show that g stabilises σ'. ∎

Corollary 1.24

$L = GL(W_{i_1}) \times \dots \times GL(W_{i_{r+1}})$ (direct product). ∎

The next two lemmas should be compared with §1.11/2/3.

<u>Lemma 1.25</u>

There is a P-equivariant bijection

$\varphi : P/L \to \{\text{decompositions of } \sigma\}$ given by $pL \to p.\Delta$.

(Here P acts on P/L by $p_1.(p_2L) = (p_1p_2)L$.)

<u>Proof</u>:

Let $p_1, p_2 \in P$. Then $p_1L = p_2L$ iff $p_2^{-1}p_1 \in L$ iff $p_2^{-1}p_1.\Delta = \Delta$ iff $p_1.\Delta = p_2.\Delta$: so φ is well-defined and injective. Moreover, $\varphi(p_1.(p_2L)) = \varphi((p_1p_2)L) = (p_1p_2).\Delta$ $= p_1.(p_2.\Delta) = p_1.\varphi(p_2L)$ so φ is P-equivariant.

Now given a decomposition $\Delta' = (W'_{i_1},\dots,W'_{i_{r+1}})$ of σ, the projection isomorphisms (see Lemma 1.21)

$$\begin{tikzcd} & W_{i_k} \arrow[d, dashed, "\cong"] \\ U_{i_k}/U_{i_{k-1}} \arrow[ur, "\delta_k", "\cong"'] \arrow[r, "\cong", "\delta'_k"'] & W'_{i_k} \end{tikzcd}$$

provide an isomorphism $\delta'_k\delta_k^{-1} : W_{i_k} \xrightarrow{\cong} W'_{i_k}$ for each k and hence an isomorphism $p' : V \xrightarrow{\cong} V$ which stabilises each $U_{i_k} = W_{i_1} \oplus \dots \oplus W_{i_k} = W'_{i_1} \oplus \dots \oplus W'_{i_k}$ so is in P. Thus $\Delta' = p'.\Delta$ with $p' \in P$. So if Δ'' is another decomposition of σ then there exists $p'' \in P$ with $\Delta'' = p''.\Delta$ and $\Delta'' = p''p'^{-1}.\Delta'$ hence P acts transitively on {decompositions of σ} and so φ is surjective. ∎

<u>Lemma 1.26</u>

(i) $\mathrm{Stab}_P(p.\Delta) = {}^{p}L \quad (p \in P)$.

(ii) If $p \in P$ and ${}^{p}L = L$ then $p \in L$. (i.e. $N_P(L) = L$).

(iii) There is a P-equivariant bijection

$\psi : \{\text{decompositions of } \sigma\} \to \{\text{Levi components of } P\}$

given by $\Delta \mapsto P \cap \mathrm{Stab}_G(\theta^{-1}(\Delta)) = \mathrm{Stab}_P(\Delta)$. (cf.§1.22/3).

(Here P acts on {Levi components of P} by conjugation, the action being well-defined by part (i)).

<u>Proof</u>:

(i) $\mathrm{Stab}_P(p.\Delta) = \{q \in P : qp.\Delta = p.\Delta\}$

$$= \{prp^{-1} : r \in P \text{ and } r.\Delta = \Delta\}$$

$$= pLp^{-1} = {}^pL.$$

(ii) Suppose $p \notin L$. Then there exists k with $1 \leq k \leq r+1$ and $p.W_{i_k} \neq W_{i_k}$. Let $U = p.W_{i_k}$. Then we shall show there exists $l \in L$ such that $l.U \neq U$, whence ${}^pL \neq L$ (using (i)). This proves (ii).

Now with respect to the basis $\underset{\sim}{e}$ of V, p may be written in block matrix form (A_{lm}) where for $1 \leq l,m \leq r+1$ A_{lm} is an $(i_l - i_{l-1}) \times (i_m - i_{m-1})$ matrix, $A_{lm} = 0$ if $l > m$, and $A_{jk} \neq 0$ for some j with $j < k$. Now there exist block diagonal matrices $l_1 = (S_l)$, $l_2 = (T_l) \in L$, with $S_l = 1_{i_l - i_{l-1}}$ for $l \neq j$, and $T_l = 1_{i_l - i_{l-1}}$ for $l \neq k$, and so that $A'_{jk} = S_j A_{jk} T_k$ has entries (a'_{pq}) with $a'_{pq} = 0$ if $p \neq q$ and at least one $a'_{pp} \neq 0$, say for $p = 1$. Since $l_1, l_2 \in L$ we may replace p by $l_1 p l_2$ and A_{jk} by A'_{jk}.

Let $l \in L$ be a diagonal matrix differing from the identity only in the $(i_{j-1}+1, i_{j-1}+1)$ - place, where the entry b is chosen so that $b \in R^*$ and $ba'_{11} \neq a'_{11}$. Then $l.e_i = e_i$ for $1 \leq i \leq n$, except for $i = i_{j-1}+1$. Thus $lp.e_i = p.e_i$ for $i_{k-1} < i \leq i_k$, except for $i = i_{k-1}+1$, and $lp.e_{i_{k-1}+1}$ is independent of $\{pe_i : i_{k-1} < i \leq i_k\}$. Hence $l.U \neq U$.

(iii) ψ is surjective since θ^{-1} (in §1.22) is bijective. Let $p_1, p_2 \in P$, and suppose $\psi(p_1.\Delta) = \psi(p_2.\Delta)$. Then by part (i) ${}^{p_1}L = {}^{p_2}L$, so ${}^{(p_2^{-1}p_1)}L = L$ and so by part (ii) $p_2^{-1}p_1 \in L$. Hence $p_2^{-1}p_1.\Delta = \Delta$, so $p_1.\Delta = p_2.\Delta$; so ψ is injective. Moreover $\psi(p_1.(p_2.\Delta)) = \psi((p_1p_2).\Delta) = {}^{(p_1p_2)}L = {}^{p_1}\psi(p_2.\Delta)$, so ψ is P-equivariant. ∎

We now have the main structural result of ~~this~~ section. First we define $U = \{p \in P$ which act as 1 on $U_{i_k}/U_{i_{k-1}}$ for $1 \leq k \leq r+1\}$ (the 'unipotent radical' of P).

Proposition 1.27

(i) $P = L.U$, semidirect product $(U \lhd P)$.

(ii) For $u \in U$ ${}^{u}L$ is a Levi component of P, and each Levi component of P has the form ${}^{u}L$ for a unique $u \in U$.

Proof:

(i)Let $1 \leq k \leq r+1$; then $\delta_k : U_{i_k} = U_{i_{k-1}} \oplus W_{i_k} \to W_{i_k}$ is L-equivariant since L stabilises both $U_{i_{k-1}}$ and W_{i_k} :

$$\delta_k(l.(v+w)) = \delta_k(lv+lw) = lw = l\delta_k(v+w) \quad (v \in U_{i_{k-1}}, w \in W_{i_k}).$$

So $\delta_k : U_{i_k}/U_{i_{k-1}} \to W_{i_k}$ is an L-equivariant isomorphism (*)

Hence if $p \in L$ and $p|U_{i_k}/U_{i_{k-1}} = 1$ for $1 \leq k \leq r+1$ then $p = 1$ since $V = W_{i_1} \oplus \ldots \oplus W_{i_{r+1}}$. So $U \cap L = 1$.

Now define $\delta : P \to L$ by $p \mapsto \prod_{k=1}^{r+1} \delta_{k*}(p)$ (cf. §1.24) , where $\delta_{k*}(p)$ is the composite isomorphism

$$W_{i_k} \xrightarrow{\delta_k^{-1}} U_{i_k}/U_{i_{k-1}} \xrightarrow{p} U_{i_k}/U_{i_{k-1}} \xrightarrow{\delta_k} W_{i_k} .$$

Then $\ker(\delta) = \{p \in P : p|U_{i_k}/U_{i_{k-1}} = 1 \ (1 \leq k \leq r+1)\} = U$, and $\delta|L = 1$, for if $p \in L$ then $\delta_{k*}(p) = p|W_{i_k}$ using (*).

So we have a short exact sequence

$$1 \to U \hookrightarrow P \overset{\delta}{\rightleftarrows} L \to 1$$

and a semidirect product $P = L.U$.

(ii) follows from Lemmas 1.25/6 and (i). ∎

Parabolic subgroups and Levi components of L

The parabolic subgroups of L are those of the form $P_1 \times \ldots \times P_{r+1}$ where P_k is a parabolic subgroup of $G_k = GL(W_{i_k})$

These are precisely the groups $Q \cap L$ where Q is a parabolic subgroup of G contained in P :

indeed, if Q has type Z, then $X \subset Z$, Q has the form $Q = \mathrm{Stab}_G(0 \subset U_{j_1} \subset \ldots \subset U_{j_s} \subset V)$, and typically $P_k = \mathrm{Stab}_{G_k}(0 \subset W_{m_1} \subset \ldots \subset W_{m_t} \subset W_{i_k})$ where, writing $X_k = \{i \in S \cup \{n\} \mid i_{k-1} < i \leq i_k\}$ we have

$$Z \cap X_k = \{m_1, \ldots m_t, i_k\} \quad \text{and}$$

and $W_{m_l} = U_{m_l} \cap W_{i_k} = U_{m_l} \cap U_{i_k} \cap U'_{n-i_{k-1}} = U_{m_l} \cap U'_{n-i_{k-1}}$.
Conversely, given such P_k, we take $U_{m_l} = U_{i_{k-1}} \oplus W_{m_l}$ to obtain Q.

A typical Levi component of P_k is

$$L_k = P_k \cap \mathrm{Stab}_{G_k}(0 \subset W'_{n-m_t} \subset \ldots \subset W'_{n-m_1} \subset W_{i_k})$$

where $W_{m_l} \cap W'_{n-m_l} = 0$ $(1 \leq l \leq t+1)$ and the Levi components of $Q \cap L = P_1 \times \ldots \times P_{r+1}$ are defined to be the subgroups of the form $L_1 \times \ldots \times L_{r+1}$.

These are also precisely the groups $M \cap L$ where M is a Levi component of Q :

indeed if $M = Q \cap Q'$ with $Q' = \mathrm{Stab}_G(0 \subset U'_{n-j_s} \subset \ldots \subset U'_{n-j_1} \subset V)$ then $W'_{n-m_l} = U'_{n-m_l} \cap W_{i_k}$; and conversely we take $U'_{n-m_l} = U_{n-i_k} \oplus W'_{n-m_l}$.

For future use we fix a standard flag

$\rho'_S = (0 \subset V'_1 \subset \ldots \subset V'_{n-1} \subset V)$ opposite to ρ_S;

then for $X \subset S$ we also have a standard flag

$\rho'_X = (0 \subset V'_{n-i_r} \subset \ldots \subset V'_{n-i_1} \subset V)$ opposite to ρ_X.

Let $P'_X = \mathrm{Stab}_G(\rho'_X)$ and let $L_X = P_X \cap P'_X$ be the <u>standard Levi component</u> of P_X: write also $T = L_S$.
Finally we write U_X for the 'unipotent radical' of P_X (§1.27).

The standard parabolic subgroups of L_X are $P_Z \cap L_X$ $(X \subset Z)$, and the corresponding standard Levi component is $L_Z \cap L_X = L_Z$.

<u>1.3 Root subgroups</u>

We let $\Delta_S = (W_1, \ldots, W_n)$ be the decomposition of ρ_S associated to the flag ρ'_S by Lemma 1.22. For each pair of integers (i,j) such that $1 \leq i,j \leq n$ and $i \neq j$ define a '<u>root subgroup</u>' :

$$U_{ij} = \{g \in U_S \mid g|W_k = 1 \text{ if } k \neq j, \text{ and } g(W_j) \subset W_i \oplus W_j\}.$$

Note that $W_i \oplus W_j = W_i \oplus g(W_j)$ for $g \in U_{ij}$. Thus U_{ij} permutes the 'lines' in $W_i \oplus W_j$ complementary to W_i and acts trivially elsewhere.

Clearly $U_{ij} \cap U_{kl} = 1$ unless $i = k,\ j = l$.

<u>Lemma 1.31</u>

$U_X = \prod U_{ij}$, the product taken over all (i,j) such that if k is such that $i_{k-1} < i \leq i_k$ then $j > i_k$.

<u>Proof</u>:

It is clear that for each k we have

$$\prod_{i \leq i_k < j} U_{ij} = \{g \in G \mid g|V_{i_k} = 1,\ g|V/V_{i_k} = 1\},$$

and the result follows. ∎

<u>Corollary 1.32</u>

Each element $u \in U_X$ has a unique expression of the form $u = \prod u_{ij}$, with $u_{ij} \in U_{ij}$, the product over (i,j) as in §1.31, taken in some arbitrary, but fixed, order. ∎

Now suppose that $\underset{\sim}{e}$ is a basis for the decomposition Δ_g, i.e. $\langle e_i \rangle = W_i$ $(1 \leq i \leq n)$.

<u>Lemma 1.33</u>

For each (i,j) with $1 \leq i,j \leq n$ and $i \neq j$ we have an isomorphism of groups : $x_{ij} : R^+ \to U_{ij}$ given by $x_{ij}(r) : \begin{cases} e_k \to e_k & (k \neq j) \\ e_j \to e_j + re_i \end{cases}$. ∎

<u>Lemma 1.34</u> (Commutator formulae).

Let $1 \leq i,j,l,m \leq n$, $i \neq j$, $l \neq m$, and let $r,s \in R$.

Then

$$\left[x_{ij}(r) \ ,\ x_{lm}(s) \right] = \begin{cases} 1 & j \neq l, i \neq m \\ x_{im}(sr) & j = l, i \neq m \\ x_{lj}(-rs) & j \neq l, i = m \end{cases}$$

There is no simple formula for this commutator if $\underline{j=l, i=m}$, but it is always equal to 1 if $rs=0$.

<u>Corollary 1.35</u>

The commutator subgroup U' of U is $\prod_{1 \leq i < j-1 \leq n-1} U_{ij}$.

§1.34/5 are well-known except possibly for the last statement of §1.34 so we indicate the proof :

Write $\delta_{km} = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$ (the Kronecker symbol). Then

$$e_k \xmapsto{x_{lm}(s)} e_k + \delta_{km} s e_l$$

$$\xmapsto{x_{ij}(r)} e_k + \delta_{kj} r e_i + \delta_{km} s (e_l + \delta_{lj} r e_i)$$

$$\xmapsto{x_{lm}(-s)} e_k + \delta_{km}(-s)e_l + \delta_{kj} r (e_i + \delta_{im}(-s)e_l) + \delta_{km} s e_l$$
$$+ \delta_{km} s \delta_{lj} r (e_i + \delta_{ir}(-s)e_l)$$

$$\xmapsto{x_{ij}(-r)} e_k + \delta_{kj}(-r)e_i + \delta_{kj} r e_i + \delta_{kj} r \delta_{ir}(-s)e_l$$
$$+ \delta_{kj} r \delta_{im}(-s) \delta_{lj}(-r) e_i + \delta_{km} s \delta_{lj} r e_i$$
$$+ \delta_{km} s \delta_{lj} r \delta_{im}(-s) e_l + \delta_{km} s \delta_{lj} r \delta_{ir}(-s) \delta_{lj}(-r) e_i$$

Hence $[x_{ij}(r) , x_{lr}(s)]$ sends e_k to

$$e_k - (rs\delta_{kj}\delta_{im} + rs^2\delta_{km}\delta_{lj}\delta_{ir})e_l$$
$$+ (rs\delta_{km}\delta_{lj} + r^2s\delta_{kj}\delta_{ir}\delta_{lj} + r^2s^2\delta_{km}\delta_{lj}^2\delta_{im})e_i$$

$$= \begin{cases} e_k & j \neq l, i \neq m \text{ or if } rs = 0 \\ e_k + \delta_{km} s r e_i & j = l, i \neq m \\ e_k - \delta_{kj} r s e_l & j \neq l, i = m \end{cases}$$ ■

Now let W be the symmetric group of permutations of $\{1,2,\ldots,n\}$. We obtain an inclusion $W \hookrightarrow G$ by defining $w(e_i) = e_{w(i)}$ for $1 \leq i \leq n$.

<u>Lemma 1.36</u>

$${}^wU_{ij} = U_{w(i),w(j)} \quad \text{for } 1 \leq i,j \leq n \text{ , } i \neq j \text{ , and } w \in W.$$

<u>Proof</u>:

$${}^wU_{ij} = \{wgw^{-1} \mid g \in G,\ g|W_k = 1\ (k \neq j),\ g(W_j) \subset W_i \oplus W_j\}$$

$= \{h \in G \mid w^{-1}hw|W_k = 1 \ (k \neq j),\ w^{-1}hw(W_j) \subset W_i \oplus W_j\}$

$= \{h \in G \mid h|w(W_k) = 1 \ (k \neq j),\ h(w(W_j)) \subset w(W_i \oplus W_j)\}$

$= \{h \in G \mid h|W_{w(k)} = 1 \ (k \neq j, \text{i.e. } w(k) \neq w(j)),$
$\qquad h(W_{w(j)}) \subset W_{w(i)} \oplus W_{w(j)}\}$

$= U_{w(i),w(j)}$. ∎

Now following [2] W is a Coxeter group generated by $\mathcal{R} = \{r_i \mid 1 \leq i \leq n-1\}$, r_i being the transposition interchanging i and $i+1$ and leaving everything else fixed. The <u>length</u> $l(w)$ of an element $w \in W$ is the number of pairs (i,j) with $1 \leq i < j \leq n$ such that $w(i) > w(j)$, and there is a unique longest element w_0 of W given by $i \mapsto n+1-i$ $(1 \leq i \leq n)$.

W is the Weyl group of a reduced, irreducible root system, of type A_{n-1}. Let $\Phi = \{\alpha_{ij} \mid 1 \leq i,j \leq n,\ i \neq j\}$ be the set of roots and $\Phi^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ the <u>positive</u> roots. The simple roots are $\Sigma = \{\alpha_{i,i+1} \mid 1 \leq i \leq n-1\}$. Also write $\alpha_i = \alpha_{i,i+1}$.

W acts on Φ by $w(\alpha_{ij}) = \alpha_{w(i),w(j)}$.

For $X = \{i_1, \ldots, i_r\} \subset S$ write $X' = S - X$. There is a subsystem $\Phi_{X'} = \{\alpha_{ij} \in \Phi \mid i_{k-1} < i \leq i_k \text{ iff } i_{k-1} < j \leq i_k \ (1 \leq k \leq r+1)\}$ and we denote $\Phi^+_{X'} = \Phi_{X'} \cap \Phi^+$.

We shall also write $-\alpha_{ij} = \alpha_{ji}$ and $\Phi^- = \{-\alpha \mid \alpha \in \Phi^+\}$ and $\Phi^-_{X'} = \Phi_{X'} \cap \Phi^-$. Finally we shall also write $U_\alpha = U_{ij}$ and $x_\alpha = x_{ij}$ for $\alpha = \alpha_{ij} \in \Phi$.

<u>Remarks</u>:

(i) We have written $\Phi_{X'}$ (and not Φ_X) to conform with the usual notation (see [2] for example).

(ii) W does not act on S and if $X \subset S$ then $\{w_0(i) \mid i \in X\}$ is not equal to $w_0(X)$ as introduced in §1.2; however if $\Sigma_X = \{\alpha_i \mid i \in X\}$ then $w_0(\Sigma_X) = -\Sigma_{w_0(X)}$: see also §1.37.

Examples 1.37

(i) $U_X = \prod_{\alpha\in\Phi^+-\Phi^+_{X'}} U_\alpha$. Also ${}^wU_\alpha = U_{w\alpha}$ by Lemma 1.36.

(ii) The 'unipotent radical' of P'_X is $U^-_X = \prod_{\alpha\in\Phi^--\Phi^-_{X'}} U_\alpha$,

since $\Delta'_S = (W_n,\ldots,W_1)$ is a decomposition of P'_S .

Also $U^-_X \cap P_X = 1$.

(iii) We have the useful relations ${}^{w_0}U_X = U^-_{w_0(X)}$,

$${}^{w_0}L_X = L_{w_0(X)} , \quad {}^{w_0}P_X = P'_{w_0(X)} .$$

(iv) For $w \in W$ we have $U \cap {}^wU^- = \prod_{\alpha>0,\, w^{-1}\alpha<0} U_\alpha$ (also denoted U^-_w), where $U = U_S$ and $U^- = U^-_S$.

1.4 Distinguished coset representatives

We begin by discussing 'distinguished coset representatives' for subgroups of parabolic type of W (cf.[5]). The results apply to arbitrary Coxeter groups.

For each $X \subset S$, W_X denotes the subgroup of W generated by $\{r_i \mid i \in X\}$: so $W_S = W$. The following is an excercise in [2] , (Chapter IV, §1).

Lemma 1.41

There is a unique element $\underline{w}$ in each coset $wW_X \in W/W_X$ of minimal length and any $w' \in wW_X$ can be written in the form $w' = \underline{w}x$ with $x \in W_X$ and $l(w') = l(\underline{w}) + l(x)$. ∎

Definition

$\underline{w}$ will be called the distinguished representative of the coset wW_X. Write $\underline{W}_X$ for the set of distinguished representatives of left cosets of W_X in W.

Lemma 1.42

Let $w \in W$. Then the following conditions are equivalent.

(i) $w \in \underline{W}_X$.

(ii) $l(w) < l(wr_i)$ for all $i \in X$.

(iii) $w(\alpha_i) > 0$ for all $i \in X$.

(iv) $w(\alpha) > 0$ for all $\alpha \in \Phi^+_X$.

Proof:

Since $W_X = < r_i \mid i \in X >$ the minimality of $l(w)$ (condition (i)) is equivalent to condition (ii).

Suppose now there exists a simple root α_i with $i \in X$ and $w(\alpha_i) < 0$. Since Φ is reduced we have $r_i(\Phi^+ - \{\alpha_i\}) = \Phi^+ - \{\alpha_i\}$ ([2],VI,§1.6,Cor.1 to Prop.17), and $l(w) = |\Phi^+ \cap w^{-1}\Phi^-|$ $= |\{\alpha \in \Phi^+ \mid w\alpha < 0\}|$ (loc.cit. Cor.2). But $wr_i(\alpha_i) =$ $w(-\alpha_i) > 0$, so $l(wr_i) = l(w) - 1 < l(w)$. Hence (ii) implies (iii); the argument may be reversed to prove the converse, since $l(wr_i)$ equals either $l(w) + 1$ or $l(w) - 1$.

(iii) and (iv) are equivalent since every $\alpha \in \Phi^+_X$ is a linear combination of the α_i $(i \in X)$ with positive coefficients, and of course $\alpha_i \in \Phi^+_X$ for $i \in X$. ∎

For the next result concerning the sets $\underline{W}_X$ (§1.45) we shall need a combinatorial proposition (§1.44) which we shall give in a general enough form for application in Chapters 2 and 4.

Let $\mathcal{S}$ be a finite non-discrete partially-ordered set. The <u>weight</u> $wt(x)$ of an element $x \in \mathcal{S}$ is the largest integer k such that there exists a chain $x_0 < x_1 < \ldots < x_k = x$ in $\mathcal{S}$. The requirement that $\mathcal{S}$ be non-discrete amounts to the existence of an element $x \in \mathcal{S}$ with $wt(x) = 1$.

We make the following assumptions about $\mathcal{S}$:

(1) For all $x,y \in \mathcal{S}$ there exists a unique $z \in \mathcal{S}$ which is maximal with the properties $z \leq x, z \leq y$. Write $z = x \cap y$.

(2) Choose $x_0, y_0 \in \mathcal{S}$ such that $y_0 \leq x_0$, and $wt(x_0) = wt(y_0)+1$. Then for all $y \in \mathcal{S}$ such that $y \leq y_0$ there exists a unique $y' \in \mathcal{S}$ such that $y' \leq x_0$ and $y = y' \cap y_0$.

Note that (1) implies that for all $x_1, \ldots, x_r \in \mathcal{S}$ the set $\{x \in \mathcal{S} \mid x \leq x_i \ (1 \leq i \leq r)\}$ has a unique maximal element (and in particular $\mathcal{S}$ has a unique minimal element).

Lemma 1.43

For all $x,y \in \mathcal{S}$ with $x \leq y_0$ and $y \leq y_0$ we have $x' \cap y' = (x \cap y)'$ and $x' \cap y = x \cap y' = x \cap y$.

Proof:

$y_0 \cap (x \cap y)' = x \cap y = (x' \cap y_0) \cap (y' \cap y_0) = y_0 \cap (x' \cap y')$, so by the uniqueness assertion in (2) we see $(x \cap y)' = x' \cap y'$. Also since $y \leq y_0$ we have $x \cap y = x' \cap y_0 \cap y = x' \cap y$ and similarly $x \cap y = x \cap y_0 \cap y' = x \cap y'$ since $x \leq y_0$. ∎

Let M be the free abelian group on a finite set F; so M consists of formal sums $a = \sum_{f \in F} a_f . f$ where $a_f \in \mathbb{Z}$ $(f \in F)$. Define M^+ by $a \in M^+$ iff $a_f \geq 0$ for all $f \in F$. If $b = \sum_{f \in F} b_f . f$ then we define $a \cap b = \sum_{f \in F} c_f . f$ where $c_f = \min(a_f, b_f)$ and we say $a \leq b$ iff $a_f \leq b_f$ for all $f \in F$.

Let $A: \mathcal{S} \to M^+$ be a function such that

$$(*) \qquad \text{for } x,y \in \mathcal{S},\ A_x \cap A_y = A_{x \cap y} \quad .$$

Then associated to A we have a function $CA: \mathcal{S} \to M$ defined by

$$CA_x = (-1)^{wt(x)} \sum_{y \leq x} (-1)^{wt(y)} A_y \qquad \text{for } x \in \mathcal{S}.$$

The combinatorial proposition alluded to above is as follows.

<u>Proposition 1.44</u>

For each $x_0 \in \Delta$: (i) CA_{x_0} lies in M^+ and is $\leq A_{x_0}$.

(ii) $A_{x_0} = \sum_{x \leq x_0} CA_x$.

<u>Proof</u>:

(i) If $wt(x_0) = 0$ then $CA_{x_0} = A_{x_0}$. So assume $wt(x_0) \geq 1$ and choose $y_0 \in \Delta$ such that $y_0 \leq x_0$ and $wt(y_0)+1 = wt(x_0)$. For each $y \in \Delta$ with $y \leq y_0$ define $B_y = A_{y'} - A_y$ (this lies in M^+); here, y' is as in (2) above. Then if $z \in \Delta$ with $z \leq y_0$ also, we have

$$\begin{aligned} B_y \cap B_z &= (A_{y'} - A_y) \cap (A_{z'} - A_z) \\ &= A_{y'} \cap (A_{z'} - A_z) - A_y \cap (A_{z'} - A_z) \\ &= A_{y'} \cap A_{z'} - A_{y'} \cap A_z - A_y \cap A_{z'} + A_y \cap A_z \quad \text{(in } M\text{)} \\ &= A_{(y \cap z)'} - A_{y \cap z} \quad \text{by Lemma 1.43 and } (*) \\ &= B_{y \cap z} . \end{aligned}$$

So $B: \Delta_{y_0} \to M^+$ satisfies $B_y \cap B_z = B_{y \cap z}$ for all $y, z \in \Delta_{y_0} = \{x \in \Delta \mid x \leq y_0\}$. Hence

$$\begin{aligned} CA_{x_0} &= (-1)^{wt(x_0)}.(-1). \sum_{y \leq y_0} (-1)^{wt(y)} (A_{y'} - A_y) \\ &= (-1)^{wt(y_0)} \sum_{y \leq y_0} (-1)^{wt(y)} B_y \end{aligned}$$

and now an obvious induction on $wt(x_0)$ shows that CA_{x_0} lies in M^+ and that $CA_{x_0} \leq B_{y_0} = A_{y_0'} - A_{y_0} = A_{x_0} - A_{y_0} \leq A_{x_0}$, proving (i) for <u>all</u> functions $A: \Delta_{x_0} \to M^+$ satisfying $A_{y \cap z} = A_y \cap A_z$ for $y, z \in \Delta_{x_0}$.

(ii) The coefficient of A_x in $\sum_{y \leq x_0} CA_y$ is 0 if $x \nleq x_0$, and if $x \leq x_0$ it is $(-1)^{wt(x)} \sum_{x \leq y \leq x_0} (-1)^{wt(y)}$. If $x = x_0$ this is 1; and if $x < x_0$ it is $(-1)^{wt(x)}.(1-1)^{wt(x_0)-wt(x)}$, by the binomial theorem, and this is 0. ∎

For the applications we shall fix positive integers h and n and take $\Delta = \{ (k_1, \ldots, k_{n-1}) \mid k_j \in \mathbb{Z}, 0 \leq k_j \leq h \ (1 \leq j \leq n-1) \}$.

<u>Proposition 1.44</u>

For each $x_0 \in \Delta$: (i) CA_{x_0} lies in M^+ and is $\leqslant A_{x_0}$.

(ii) $A_{x_0} = \sum_{x \leqslant x_0} CA_x$.

<u>Proof</u>:

(i) If $wt(x_0) = 0$ then $CA_{x_0} = A_{x_0}$. So assume $wt(x_0) \geq 1$ and choose $y_0 \in \Delta$ such that $y_0 \leqslant x_0$ and $wt(y_0)+1 = wt(x_0)$. For each $y \in \Delta$ with $y \leqslant y_0$ define $B_y = A_{y'} - A_y$ (this lies in M^+); here, y' is as in (2) above. Then if $z \in \Delta$ with $z \leqslant y_0$ also, we have

$$
\begin{aligned}
B_y \cap B_z &= (A_{y'} - A_y) \cap (A_{z'} - A_z) \\
&= A_{y'} \cap (A_{z'} - A_z) - A_y \cap (A_{z'} - A_z) \\
&= A_{y'} \cap A_{z'} - A_{y'} \cap A_z - A_y \cap A_{z'} + A_y \cap A_z \quad \text{(in } M\text{)} \\
&= A_{(y \cap z)'} - A_{y \cap z} \qquad \text{by Lemma 1.43 and } (*) \\
&= B_{y \cap z} .
\end{aligned}
$$

So $B: \Delta_{y_0} \to M^+$ satisfies $B_y \cap B_z = B_{y \cap z}$ for all $y, z \in \Delta_{y_0} = \{x \in \Delta \mid x \leqslant y_0\}$. Hence

$$
\begin{aligned}
CA_{x_0} &= (-1)^{wt(x_0)}.(-1). \sum_{y \leqslant y_0} (-1)^{wt(y)} (A_{y'} - A_y) \\
&= (-1)^{wt(y_0)} \sum_{y \leqslant y_0} (-1)^{wt(y)} B_y
\end{aligned}
$$

and now an obvious induction on $wt(x_0)$ shows that CA_{x_0} lies in M^+ and that $CA_{x_0} \leqslant B_{y_0} = A_{y_0'} - A_{y_0} = A_{x_0} - A_{y_0} \leqslant A_{x_0}$, proving (i) for <u>all</u> functions $A: \Delta_{x_0} \to M^+$ satisfying $A_{y \cap z} = A_y \cap A_z$ for $y, z \in \Delta_{x_0}$.

(ii) The coefficient of A_x in $\sum_{y \leqslant x_0} CA_y$ is 0 if $x \nleqslant x_0$, and if $x \leqslant x_0$ it is $(-1)^{wt(x)} \sum_{x \leqslant y \leqslant x_0} (-1)^{wt(y)}$. If $x = x_0$ this is 1; and if $x < x_0$ it is $(-1)^{wt(x)}.(1-1)^{wt(x_0)-wt(x)}$, by the binomial theorem, and this is 0. ∎

For the applications we shall fix positive integers h and n and take $\Delta = \{ (k_1, \ldots, k_{n-1}) \mid k_j \in \mathbb{Z}, 0 \leqslant k_j \leqslant h \ (1 \leqslant j \leqslant n-1) \}$.

The partial order on $\mathcal{L}$ is given by $(k_1,\dots,k_{n-1}) \leq (l_1,\dots,l_{n-1})$ iff $k_j \leq l_j$ $(1\leq j\leq n-1)$. So we have

(1) $(k_1,\dots,k_{n-1}) \cap (l_1,\dots,l_{n-1}) = (m_1,\dots,m_{n-1})$

where $m_j = \min(k_j,l_j)$ $(1\leq j\leq n-1)$.

(2) $wt(k_1,\dots,k_{n-1}) = \sum_{j=1}^{n-1} k_j$.

So if $x_0 = (k_1,\dots,k_{n-1})$ then for some j we have $y_0 = (k_1,\dots,k_{j-1},k_j-1,k_{j+1},\dots,k_{n-1})$ and if $y \leq y_0$, say $y = (l_1,\dots,l_{n-1})$ then $y' = (l_1,\dots,l_j+1,\dots,l_{n-1})$.

Special case

When $h = 1$ we may identify $\mathcal{L} = \{$subsets of $S=\{1,2,\dots,n-1\}$ $\}$, partially-ordered by inclusion. Then if $x,y \in \mathcal{L}$ $x\cap y$ is just the set-theoretic intersection and $wt(x) = |x|$; $x_0 = y_0 \cup \{s\}$ for some $s \in S$ and if $y \leq y_0$ then $y' = y \cup \{s\}$.

We shall apply §1.44 to two types of situation.

Case (a)

$F = \{$singleton subsets of a finite set $Y\}$. Then we may identify the subsets $Z \subset Y$ as elements of M^+ via $Z = \sum_{f\subset Z} f$, the sum of the singleton subsets of Z. We may take $A:\mathcal{L} \to M^+$ so that for $x \in \mathcal{L}$ A_x is a subset of Y and then $A_x \cap A_y$ is just the set-theoretic intersection.

Case (b)

$F = \{$irreducible components of a given decomposition of a finite-dimensional representation Y of $G\}$. Then we may identify as elements $Z = \sum_{f\subset Z} f$ of M^+ those subrepresentations Z of Y which decompose into a sum of elements of F. A is taken so that A_x is such a subrepresentation of Y, and then $A_x \cap A_y$ is just the sum of the components (from F) contained in both A_x and A_y.

We wish to apply §1.44 with $A_X = \underline{W}_{X'}$ $(X \subset S)$. $(h=1, \text{Case (a)})$.

So we must check condition $(*)$: for $X, Y \subset S$ we have

$$\begin{aligned} \underline{W}_{X'} \cap \underline{W}_{Y'} &= \{w \in W \mid w(\alpha_i) > 0 \text{ for all } i \in X' \text{ and } i \in Y'\} \\ &= \{w \in W \mid w(\alpha_i) > 0 \text{ for all } i \in X' \cup Y' = (X \cap Y)'\} \\ &= \underline{W}_{(X \cap Y)'}. \end{aligned}$$

We write C_X for $C\underline{W}_{X'}$. Then $\underline{W}_{X'}$ is the disjoint union $\bigcup_{Y \subset X} C_Y$, and $W = \underline{W}_{\emptyset} = \bigcup_{Y \subset S} C_Y$. Hence

$$C_X = \{w \in \underline{W}_{X'} \mid \text{if } Y \subsetneqq X \text{ then } w \notin \underline{W}_{Y'}\}.$$

__Corollary 1.45__

Let $w \in W$. Then $w \in C_X$ if and only if : $w(\alpha_i) > 0$ iff $i \in X'$.

In particular we have $C_S = \{w_0\}$ (a result of [5], §3).

__Proof__:

Using §1.44,

$$\begin{aligned} w(\alpha_i) > 0 \text{ iff } i \in X' &\iff w \in \underline{W}_{X'} \text{ and if } Y \subsetneqq X \text{ there exists} \\ &\qquad i \in Y' - X' \text{ such that } w(\alpha_i) < 0 \\ &\iff w \in \underline{W}_{X'} \text{ and if } Y \subsetneqq X \text{ then } w \notin \underline{W}_{Y'} \\ &\iff w \in C_X. \qquad \blacksquare \end{aligned}$$

We conclude this section by giving distinguished coset representatives for a parabolic subgroup of G (compare [5], §2).

__Theorem 1.46__

Let $X \subset S$ and $\mathfrak{X} = \{uw \mid w \in \underline{W}_{X'},\ u = \prod_{w^{-1}\alpha \in \Phi^- - \Phi^-_{X'}} x_\alpha(r_\alpha),$ where $r_\alpha \in R$ and $\pi \mid r_\alpha$ if $\alpha < 0\}$. Then

(i) $|\mathfrak{F}_X| = |\mathfrak{X}|$.

(ii) If $g_1, g_2 \in \mathfrak{X}$ then $g_1 \cdot \rho_X = g_2 \cdot \rho_X$ iff $g_1 = g_2$

(iii) $\mathfrak{F}_X = \{g \cdot \rho_X \mid g \in \mathfrak{X}\}$.

Thus each element $\sigma \in \mathfrak{F}_X$ has a unique representative $g \in \mathfrak{X}$ such that $\sigma = g.\rho_X$ which will be called its <u>distinguished</u> representative in G. By §1.11 $\mathfrak{X}$ is also a set of representatives for left cosets of P_X in G.

If $g = uw \in \mathfrak{X}$ then we shall say that $\sigma = g.\rho_X$ <u>lies over</u> w; denote by $\mathfrak{F}_X(w)$ the set of such σ. Note that $u \in {}^wU_X^-$.

In proving §1.46 we shall need some results for the case when R is a field; so <u>until after §1.49 assume that h = 1</u>.

<u>Bruhat decomposition</u>

Given flags $\sigma = (0 \subset U_1 \subset \ldots U_{n-1} \subset V)$ and $\tau = (0 \subset W_1 \subset \ldots \subset W_{n-1} \subset V) \in \mathfrak{F}_S$ define $w \in W$ as follows :

suppose $i \leq n$ and $w(j)$ $(1 \leq j \leq i-1)$ are already defined; then $w(i)$ is given by $w(i) \neq w(j)$ $(1 \leq j \leq i-1)$ and

$$U_i \cap W_{w(i)-1} \neq U_i \cap W_{w(i)} .$$

In this case write $\tau \xrightarrow{w} \sigma$. ('$\sigma$ is in position w relative to τ').

More generally, given $\sigma_Y = (0 \subset U_{j_1} \subset \ldots \subset U_{j_s} \subset V) \in \mathfrak{F}_Y$ and $\tau_X = (0 \subset W_{i_1} \subset \ldots \subset W_{i_r} \subset V) \in \mathfrak{F}_X$ then choose subspaces U_j, W_i of V, free of rank j,i respectively ($j \in Y', i \in X'$) such that $U_{j-1} \subset U_j, W_{i-1} \subset W_i$ $(1 \leq i,j \leq n)$, obtaining flags of type S: $\sigma' = (0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$, $\tau' = (0 \subset W_1 \subset \ldots \subset W_{n-1} \subset V)$. Then there exists a unique $w \in W$ such that $\tau' \xrightarrow{w} \sigma'$.

<u>Theorem 1.47</u> (cf.[2] ,Chap.IV,§2)

Taking $\tau_X = \rho_X$ the above construction induces a bijection:

$$P_X\text{-orbits on } \mathfrak{F}_Y \longleftrightarrow \text{double cosets } W_{X'}\backslash W/W_{Y'} . \quad \blacksquare$$

In case X = Y = S we have the following improvement:

<u>Theorem 1.48</u> ([4],§3.4)

Any $g \in G$ has a unique expression uwb with $w \in W, u \in U_w^-, b \in B$. ∎

§1.46 generalises §1.48. However it will be one of the aims of Chapter 5 to show that there can be no such simple geometric description of the B-orbits on $\mathfrak{F}_S$ as that given in §1.47, when R is not a field, except in case n = 2.

Corollary 1.49

(i) $|G| = |B|.\sum_{w\in W}|U_w^-| = |B|.\sum_{w\in W} q^{l(w)}$ ([18],§9).

(ii) $|P_X| = |B|.\sum_{w\in W_{X'}} q^{l(w)}$ for $X \subset S$.

(iii) $|G/P_X| = |\mathfrak{F}_X| = \sum_{\underline{w}\in \underline{W}_{X'}} q^{l(w)}$. (cf.[5],§3.4)

Proof:

(i) As remarked before $|\Phi^+ \cap w\Phi^-| = l(w^{-1}) = l(w)$, so

$$|U_w^-| = |R^+|^{l(w)} = q^{l(w)}.$$

(ii) $|P_X| = |U_X|.|L_X|$ (§1.27)

$= |U_X|.\prod_{k=1}^{r+1}|G_k|$ (§1.24)

$= |U_X|.\prod_{k=1}^{r+1}|B_k|.\sum_{w\in S(k)} q^{l(w)}$ (induction)

where $S(k)$ = {permutations on X_k}, $B_k = G_k \cap P_{X_k} = G_k \cap B$ (§1.2).
So $|P_X| = |U_X|.|L_X \cap B|.\sum_{w\in W_{X'}} q^{l(w)}$ since $W_{X'} = S(1)\times\ldots\times S(r+1)$.

$$= |B|.\sum_{w\in W_{X'}} q^{l(w)}.$$

(iii) The unique expression of $w \in W$ as $w = \underline{w}x$ with $\underline{w} \in \underline{W}_{X'}$, $x \in W_{X'}$ and $l(w) = l(\underline{w}) + l(x)$ implies

$$\sum_{w\in W} q^{l(w)} = \sum_{\underline{w}\in \underline{W}_{X'}} q^{l(\underline{w})} . \sum_{x\in W_{X'}} q^{l(x)} .$$ ∎

Proof of §1.46:

First note that the map $\pi_1 : R \to R/\mathfrak{m} = \overline{R}$ of §1.0 induces maps $M \to M/\mathfrak{m}.M$ for any R-module M, and $\mathfrak{F}_X \to \overline{\mathfrak{F}}_X$, (given by $(0 \subset U_{i_1} \subset \ldots \subset U_{i_r} \subset V) \mapsto (0 \subset \overline{U}_{i_1} \subset \ldots \subset \overline{U}_{i_r} \subset \overline{V})$). And for any subgroup H of G we have $H \to \overline{H}$, whose kernel will be denoted by $H(\mathfrak{m})$ (cf.[9],§2.2).

Now $|\mathfrak{F}_X|/|\overline{\mathfrak{F}}_X| = (|G|/|P_X|)/(|\overline{G}|/|\overline{P_X}|)$ (§1.11)

$= (|G|/|\overline{G}|)/(|P_X|/|\overline{P_X}|)$

$= |G(\mathfrak{m})|/|P_X(\mathfrak{m})|,$

and with respect to the basis $\underset{\sim}{e}$ we have matrix forms

$G(\mathfrak{m}) = \{(a_{ij}) \mid a_{ij}\in\mathfrak{m}$ if $i\neq j$, $a_{ii}\in R^{(1)}$, $(1\leq i,j\leq n)\}$

$P_X(\mathfrak{m}) = \{(a_{ij}) \mid a_{ij}\in\mathfrak{m}$ if $i\neq j$, $a_{ij}=0$ if $i_{k-1}<j\leq i_k<i$ for some k, $a_{ii}\in R^{(1)}$ $(1\leq i,j\leq n)\}$

$U_X^-(\mathfrak{m}) = \{(a_{ij}) \mid a_{ij}\in\mathfrak{m}$ if $i\neq j$, $a_{ij}=0$ except when there exists k with $i_{k-1}<j\leq i_k<i$, $a_{ii}=1$, $(1\leq i,j\leq n)\}$.

Hence $|\mathfrak{F}_X|/|\overline{\mathfrak{F}}_X| = q^{n^2(h-1)}/q^{(|\Phi^+\cup\Phi_X^-|+n)(h-1)}$

$= q^{(|\Phi^- - \Phi_X^-|)(h-1)}$

$= |U_X^-(\mathfrak{m})|.$

But if $y\in\overline{\mathfrak{X}} = \{vw \mid w\in \underline{W}_{X'}, v\in \overline{U_w^-}\}$ and y lies over w then

$|\{x\in\mathfrak{X} \mid \overline{x} = y\}| = |{}^wU_X^-(\mathfrak{m})| = |U_X^-(\mathfrak{m})|$ and so

$|\mathfrak{X}| = |U_X^-(\mathfrak{m})|.|\overline{\mathfrak{X}}| = |U_X^-(\mathfrak{m})|.|\overline{\mathfrak{F}}_X|$ by §1.49(iii) and §1.11

$= |\mathfrak{F}_X|$. So (i) is proved.

Now suppose $g_1 = u_1w_1$, $g_2 = u_2w_2 \in\mathfrak{X}$ and $g_1.\rho_X = g_2.\rho_X$. Then

$\overline{u_1}w_1.\overline{\rho_X} = \overline{g_1}.\overline{\rho_X} = \overline{g_1.\rho_X} = \overline{g_2.\rho_X} = \overline{u_2}w_2.\overline{\rho_X}$, so by §1.48 $w_1 = w_2$.

Hence $g_2^{-1}g_1 = w_1^{-1}u_2^{-1}u_1w_1 \in w_1^{-1}(w_1U_X^-w_1^{-1})w_1 = U_X^-$.

But if $g_1.\rho_X = g_2.\rho_X$ then $g_2^{-1}g_1.\rho_X = \rho_X$ and so $g_2^{-1}g_1 \in P_X$.

Hence $g_2^{-1}g_1 \in U_X^-\cap P_X = 1$, so $g_1 = g_2$, proving (ii).

(iii) follows from (i) and (ii). ∎

<u>Remarks</u>

We have $|\overline{B}| = (q-1)^n q^{n(n-1)/2}$ so by §1.49 we have

$$|P_X| = |\overline{P_X}|.|P_X(\mathfrak{m})| = (q-1)^n \sum_{w\in W_{X'}} q^{l(w)}.\ q^{n(n-1)/2} \times q^{(n(n+1)/2 + |\Phi_X^+|)(h-1)}$$

$$|\mathfrak{F}_X| = \sum_{w\in \underline{W}_{X'}} q^{l(w)}.\ q^{(|\Phi^+ - \Phi_X^+|)(h-1)} \qquad (|\Phi^+| = n(n-1)/2)$$

~~Finally,~~ since $\sum_{w\in W} q^{l(w)} = \prod_{i=1}^{n-1}(1+q+q^2+\ldots+q^i)$ ([18], pp.132/3)

we have $$|G| = \prod_{i=1}^{n} q^{i-1}(q^i-1).\ q^{n^2(h-1)}.$$

1.5 Semisimple and regular elements

Suppose that R' is an unramified extension of R and let $V' = V \otimes_R R'$. The Frobenius map (§1.02) $F: R' \to R'$, such that $R = R'^F$, induces an automorphism $F: V' \to V'$ given by

$$F\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} F(a_i) e_i \qquad (a_i \in R', 1 \leq i \leq n) \qquad (*)$$

where $\underset{\sim}{e} = \{e_1, \ldots, e_n\}$ is a free basis of V'; thus $F(e_i) = e_i$ so $\underset{\sim}{e}$ is also a basis of V'^F and we identify $V = V'^F$. (Then $(*)$ is independent of the particular choice of basis of V).

Writing $G' = GL(V')$ we have an automorphism $F: G' \to G'$ with $G = G'^F$ (cf. [9], §3.1.4). Thus the G-action on V' commutes with the F-action : $F\left(g\left(\sum_{i=1}^{n} a_i e_i\right)\right) = F\left(\sum_{i=1}^{n} a_i . g e_i\right) = \sum_{i=1}^{n} F(a_i) g e_i$

$= g\left(\sum_{i=1}^{n} F(a_i) e_i\right) = g\left(F\left(\sum_{i=1}^{n} a_i e_i\right)\right) \qquad (a_i \in R', g \in G).$

Definition

The element $t \in G$ is <u>unramified semisimple</u> if there is an unramified extension R' of R and a decomposition of V' into free rank 1 submodules, $V' = W_1 \oplus \ldots \oplus W_n$, with each W_i stabilised by t (regarded as an element of G').

In this case t is said to <u>split over R'</u>; t is <u>split</u> if it splits over R. Note that t acts as scalar on each W_i, so there is $t_i \in R'$ such that $t(v) = t_i v$ $(v \in W_i)$ - in fact $t_i \in R'^*$, since t is non-singular.

Lemma 1.51

Let $V' = \bigoplus_{i=1}^{n} W_i = \bigoplus_{i=1}^{n} U_i$ be decompositions of V' into free rank 1 submodules W_i, U_i and suppose $t \in G'$ stabilises each W_i and each U_i, i.e. there exist $t_i, s_i \in R'^*$ with $t(v) = t_i v$ if $v \in W_i$ and $t(v) = s_i v$ if $v \in U_i$ $(1 \leq i \leq n)$.

Then there exists $w \in W$ such that $t_i = s_{w(i)}$ $(1 \leq i \leq n)$.

This lemma shows that the set $\Lambda = \{t_1, \ldots t_n\}$ of eigenvalues of t is a canonical invariant of t. (It holds for any commutative local ring R'; bars denote reduction modulo the maximal ideal, as before).

Proof:

Choose $e_i \in W_i, f_i \in U_i$ $(1 \leq i \leq n)$ so that $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are free bases of V'. By induction we shall show

(*) Let $0 \leq k \leq n$. Then there exist distinct integers $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$ so that $t_r = s_{j_r}$ $(1 \leq r \leq k)$ and so that $V' = < f_{j_1}, \ldots, f_{j_k}, e_{k+1}, \ldots, e_n >$.

The induction starts with $k = 0$, trivially, so fix m with $1 \leq m \leq n$ and assume (*) is true for $k = m-1$.

Then there exists $j_m \in \{1, \ldots, n\}$ so that $j_m \neq j_r$ $(1 \leq r \leq m-1)$ and

$$\overline{f}_{j_m} \notin < \overline{f}_{j_1}, \ldots, \overline{f}_{j_{m-1}}, \overline{e}_{m+1}, \ldots, \overline{e}_n > \qquad (**),$$

since otherwise $\overline{V'} = < \overline{f}_1, \ldots, \overline{f}_n >$ would be contained in the $(n-1)$-dimensional space $< \overline{f}_{j_1}, \ldots, \overline{f}_{j_{m-1}}, \overline{e}_{m+1}, \ldots, \overline{e}_n >$, a contradiction; and by inductive hypothesis there exist $a_i \in R'$ with $f_{j_m} = \sum_{i=1}^{m-1} a_i f_{j_i} + \sum_{i=m}^{n} a_i e_i$. In fact $a_m \in R'^*$, since otherwise (**) would be contradicted.

$$\text{Now} \quad t(f_{j_m}) = \sum_{i=1}^{m-1} a_i t(f_{j_i}) + \sum_{i=m}^{n} a_i t(e_i)$$

$$= \sum_{i=1}^{m-1} a_i t_i f_{j_i} + \sum_{i=m}^{n} a_i t_i e_i \quad \text{using (*) for } k=m-1$$

But also

$$t(f_{j_m}) = s_{j_m} f_{j_m} = \sum_{i=1}^{m-1} s_{j_m} a_i f_{j_i} + \sum_{i=m}^{n} s_{j_m} a_i e_i .$$

So since $\{f_{j_1}, \ldots, f_{j_{m-1}}, e_m, \ldots, e_n\}$ forms a basis of V' we see $s_{j_m} a_m = a_m t_m$, whence $s_{j_m} = t_m$ since $a_m \in R'^*$. Moreover

$$e_m = a_m^{-1} \left(\sum_{i=1}^{m-1} (-a_i) f_{j_i} + f_{j_m} + \sum_{i=m+1}^{n} (-a_i) e_i \right), \text{ so}$$

$V' = < f_{j_1}, \ldots, f_{j_{m-1}}, f_{j_m}, e_{m+1}, \ldots, e_n >$, and the induction continues. ∎

Let $\xi \in R'^*$ be an eigenvalue of t and let $E_\xi = \{v \in V' : t.v = \xi v\}$; this is an R'-submodule of V'.

Now if $v \in E_\xi$ then $t(Fv) = F(tv) = F(\xi v) = F\xi.Fv$, and so $F\xi$ is also an eigenvalue of t, and $E_{F\xi} = F(E_\xi)$; more generally we have $E_{F^i\xi} = F^i(E_\xi) \quad (0<i\in \mathbb{Z})$.

Suppose $\eta \in R'^*$ is another eigenvalue of t $(\eta \neq \xi)$ and suppose $v \in V'$ has order π^h, i.e. $\pi^{h-1}v \neq 0$. Then $(\eta-\xi)v \neq 0$, i.e. $\eta v \neq \xi v$, and so $v \notin E_\eta \cap E_\xi$. (Hence $\overline{E}_\eta \cap \overline{E}_\xi = 0$).

Choose f minimally so that $F^f\xi = \xi$; then $\xi, F\xi, \ldots, F^{f-1}\xi$ are distinct. Hence we may choose $v \in E_\xi$ so that the R'-module $M = < v, Fv, \ldots, F^{f-1}v >$ is F-stable (in fact $F^f v = v$) and free of rank f over R'.

The minimality of f ensures that t fixes no F-stable free submodule of M except M and 0, so t acts <u>anisotropically</u> on the free R-submodule M^F of V of rank f (i.e. t fixes no proper non-trivial free R-submodule of M^F).

Thus writing Λ as the <u>union</u> $\bigcup_{i=1}^{r} \Lambda_i$ of (minimal) F-stable subsets $\Lambda_i = \{\xi_i, F\xi_i, \ldots, F^{f_i-1}\xi_i\}$ we obtain integers $f_1, \ldots, f_r$ with $f_1+\ldots+f_r = n$, and a decomposition $V = M_1^F \oplus \ldots \oplus M_r^F$ with t acting anisotropically on each M_i^F.

<u>Corollary 1.52</u>

There exists a group L which is minimal for the property of being a Levi subgroup of G containing t, and all such minimal subgroups are conjugate.

<u>Proof</u>:

Take $L = \prod_{i=1}^{r} GL(M_i^F)$. The conjugacy class of L is given by the partition $(f_1, \ldots, f_r)$ of n, canonically determined since Λ is so. ∎

Now write $X = \{i_1,\ldots,i_{r-1}\}$ where $i_j = \sum_{k=1}^{j} f_k$ and assume that $\underline{e} = \{e_1,\ldots,e_n\}$ is a fixed basis of V such that $e_i \in M_j^F$ for $i_{j-1} < i \leq i_j$ $(1 \leq j \leq r)$. Then $L = L_X$.

Lemma 1.53

(i) If $w \in W$ and $w^{-1}tw \in P_Y$ for some $Y \subset S$ then $w^{-1}tw \in L_Y$; in fact $w^{-1}L_Xw \subset L_Y$.

(ii) Suppose $Y \subset S$ and L_Y is conjugate to L_X. Let $w \in C_Y$. Then $w^{-1}tw \in L_Y$ iff $Y = w_0(X)$ and $w = w_0w_{Y'}$.

(Here w_Z denotes the longest element of W_Z $(Z \subset S)$.)

Proof:

(i) Let $Y = \{j_1,\ldots,j_s\}$. Then $w^{-1}tw$ stabilises each of the R-modules $< e_1,\ldots,e_{j_m} >$ $(1 \leq m \leq s+1)$ and so t stabilises each $U_m = < e_{w(1)},\ldots,e_{w(j_m)} >$ $(1 \leq m \leq s+1)$. But there exists j such that $1 \leq j \leq r$ and $e_{w(1)} \in M_j^F$; so since t acts anisotropically on each M_j^F we see $e_i \in U_m$ $(i_{j-1} < i \leq i_j)$ and $M_j^F \subset U_m$. It follows that each U_m is a sum of modules M_j^F, and hence each $< e_{w(j_{m-1}+1)},\ldots,e_{w(j_m)} >$ $(1 \leq m \leq s+1)$ is a sum of modules M_j^F, so is stabilised by t.

Hence $w^{-1}tw$ stabilises each $< e_{j_{m-1}+1},\ldots,e_{j_m} >$; thus $w^{-1}tw \in L_Y$. In fact each $< e_{j_{m-1}+1},\ldots,e_{j_m} >$ is a sum of modules $w^{-1}(M_j^F)$, so $w^{-1}L_Xw = \{l \in G : wlw^{-1}$ stabilises each $M_j^F\} = \{l \in G : l(w^{-1}M_j^F) = w^{-1}M_j^F$ for each $j\}$ is contained in L_Y.

(ii) If L_Y is conjugate to L_X then $|L_Y| = |L_X|$; but $L_X \subset {}^wL_Y$ by (i) and so $L_X = {}^wL_Y$. Hence ${}^{w_0w}L_Y = {}^{w_0}L_X = L_{w_0(X)}$ (§1.37).

Then given m, $w_0w\{i : j_{m-1} < i \leq j_m\} = w_0\{i : i_{k-1} < i \leq i_k\} = \{i : n-i_k < i \leq n-i_{k-1}\}$, for some k. w_0w acts as an element of $W_{Y'}$, which stabilises the blocks $\{i : j_{m-1} < i \leq j_m\}$, followed by a permutation of these blocks.

But if $w \in C_Y$ then $w_0w(i) > w_0w(i+1)$ iff $i \in Y'$ (§1.45), so $w_0w(j_m) < w_0w(j_m+1)$ $(1 \leq j \leq s)$ which shows the blocks must remain in the same order. Hence ${}^{w_0w}L_Y = L_Y$ and $w_0w \in W_{Y'}$.

So $Y = w_0(X)$ and $w = w_0w_{Y'}$, since $w \in \underline{W}_{Y'}$ and w_0 is the longest element of W. The converse is now also clear. ∎

Definition

For $1 \leq i, j \leq n$ $(i \neq j)$ define $\alpha_{ij}(t) = t_i t_j^{-1} \in R'^*$ and define $k_{ij}(t)$ to be the largest integer k such that $\alpha_{ij}(t) \in R'^{(k)}$.

The following properties are immediate.

Lemma 1.54

(i) $\alpha_{ij}(t^{-1}) = \alpha_{ij}(t)^{-1} = \alpha_{ji}(t)$

(ii) $k_{ij}(t) = k_{ij}(t^{-1}) = k_{ji}(t)$

(iii) $k_{ij}(t) = k$ iff $(1-\alpha_{ij}(t)) \| \pi^k$ iff $(t_i - t_j) \| \pi^k$. ∎

For $\alpha = \alpha_{ij} \in \Phi$ we shall often denote $\alpha_{ij}(t)$ by $\alpha(t)$ and $k_{ij}(t)$ by $k_\alpha(t)$; the notation is motivated by the usual identification (in the context of reductive groups over an algebraically closed field, [1], §3) of roots as characters of a maximal torus, and in fact $k_\alpha(t)$ plays the role of $\mathrm{val}(t^\alpha - 1)$ in [9], §4.4.7.

The integers $k_\alpha(t)$ depend on an ordering of the eigenvalues of t; henceforth we shall assume that, with $X = \{i_1, \ldots, i_{r-1}\}$ as above, and L_X minimal for being a Levi subgroup of G containing t, each of the sets $\Lambda_j = \{t_i : i_{j-1} < i \leq i_j\}$ $(1 \leq j \leq r)$ is F-stable. Then the ordering of the t_i is well-defined up to a permutation given by an element $w_1 \in W_X$, followed by possible interchanging of sets Λ_j which are of equal size.

But $w_1\{\alpha_{lm} : i_{j-1}<l<m\leq i_j\}$ contains exactly one of α_{lm}, α_{ml} for each (l,m) with $i_{j-1}<l<m\leq i_j$, and $k_{lm}(t) = k_{ml}(t)$, so we see $\sum_{\alpha\in\Phi^+_{X'}} k_\alpha(t)$ depends only on t and X.

More generally, let $w \in W$ and $Y \subset S$ be such that $w^{-1}tw \in P_Y$. Then arguing as in §1.53(i), with $Y = \{j_1,\ldots,j_s\}$ we see that each set $\{t_i : j_{m-1}<i\leq j_m\}$ is a union of sets $\{t_i : i_{j-1}<i\leq i_j\}$.

<u>Corollary 1.55</u>

$\sum_{\alpha\in\Phi^+_{Y'}} k_\alpha(w^{-1}tw)$ is a canonical invariant of t,w and Y.

For each $\alpha\in\Phi$ we write U'_α for the subgroup of G' defined in the same way as the subgroup U_α of G in §1.3. We identify U_α with $(U'_\alpha)^F$ and we have an isomorphism $x_\alpha : R'^+ \to U'_\alpha$ which commutes with F, i.e. $(U'_\alpha)^F = x_\alpha(R')^F = x_\alpha(R'^F) = x_\alpha(R) = U_\alpha$.

<u>Lemma 1.56</u>

(i) $tx_\alpha(r)t^{-1} = x_\alpha(\alpha(t).r)$ for $r \in R'$.

(ii) $[t,x_\alpha(r)] = x_\alpha(r(1-\alpha(t^{-1})))$ for $r \in R'$

$= 1$ iff $r \in \pi^{h-k}R'$ $(k=k_\alpha(t))$.

(iii) If $c \in (L_X)'$ lies over $1 \in W_{X'}$, and $u = \prod x_\alpha(r_\alpha) \in U_X$ then if L_X is minimal for the property of being a Levi subgroup of G containing t, as above, we have

$$[u,c^{-1}tc] = 1 \text{ iff } \alpha(t).r_\alpha = r_\alpha \text{ for all } \alpha\in\Phi^+-\Phi^+_{X'}.$$

(there is a similar result if $u \in U^-_X$).

<u>Proof</u>:

(i) $tx_\alpha(r)t^{-1}$ acts as 1 on W_k $(k\neq j)$, where $\alpha=\alpha_{ij}$, since $x_\alpha(r)$ does so and t,t^{-1} are inverse isomorphisms stabilising each W_k. On W_j we have $e_j \xmapsto{t^{-1}} t_j^{-1}e_j \xmapsto{x_\alpha(r)} t_j^{-1}(e_j + re_i)$ $\xmapsto{t} t_j^{-1}.t_je_j + t_j^{-1}r.t_ie_i = e_j + \alpha(t)re_i$.

(ii) The first equation is immediate from (i). The second equation follows from §1.54.

(iii) By §1.46 we may write c in the form $c = tv_1v_2$ with $t \in (T)'$, $v_1 \in (U^- \cap L_X)'$, $v_2 \in (U \cap L_X)'$. Write $v_1v_2 = \prod_{\beta \in \Phi_{X'}} x_\beta(r_\beta)$. Now if $X = \{i_1,\ldots,i_r\}$ then $\Phi^+ - \Phi^+_{X'}$ is the disjoint union of 'blocks' $B_{lm} = \{\alpha_{ij} : i_{l-1}<i\leq i_l \leq i_{m-1}<j\leq i_m\}$ $(1\leq l<m\leq r+1)$. If $\alpha \in B_{lm}$ and $\beta \in \Phi_{X'}$ then $\beta \neq -\alpha$, so by §1.34, $x_\beta(r_\beta)x_\alpha(r_\alpha)x_\beta(-r_\beta) = x_\alpha(r_\alpha)x_\gamma(\delta r_\alpha r_\beta)$ where $\delta = 0, 1,$ or -1 and $\gamma \in B_{lm}$. So since $(T)'$ normalises each U' (part (i)) we have ${}^cu = \prod x_\alpha(r'_\alpha)$ where the r'_α $(\alpha \in B_{lm})$ are linear combinations of the r_α $(\alpha \in B_{lm})$ and vice versa ------(*).

Hence $[u,c^{-1}tc] = 1$ iff $t.{}^cu.t^{-1} = {}^cu$

iff $\prod t x_\alpha(r'_\alpha)t^{-1} = \prod x_\alpha(r'_\alpha)$

iff $\prod x_\alpha(\alpha(t)r'_\alpha) = \prod x_\alpha(r'_\alpha)$

iff $\alpha(t)r'_\alpha = r'_\alpha$ for all $\alpha \in \Phi^+ - \Phi^+_{X'}$,

by §1.36; but using (*) and the fact that the minimality of L_X implies that $k_\alpha(t)$ is the same for all α in a given block, we see the last condition is equivalent to $\alpha(t)r_\alpha = r_\alpha$ for all $\alpha \in \Phi^+ - \Phi^+_{X'}$. ∎

This completes the list of technical results on semisimple elements required for chapter 2. We conclude this section by discussing regular elements, in particular obtaining a criterion for an unramified semisimple element to be regular.

<u>Regular elements</u>

The following definition is well-known and may be made for R an arbitrary commutative ring with 1.

An endomorphism $t:V \to V$ is <u>regular</u> iff there exists $v \in V$ which is a <u>cyclic vector</u> for t, i.e. $\{v, tv, \ldots, t^{n-1}v\}$ forms a free basis for V.

In this case $t^n v = \sum_{i=0}^{n-1} a_i t^i v$ for some $a_i \in R$ and so with respect to the basis $\{v, tv, \ldots, t^{n-1}v\}$ of V t has the 'companion matrix' form

$$C(t) = \begin{pmatrix} 0 & 0 & \cdots & \cdot & a_0 \\ 1 & 0 & \cdots & \cdot & a_1 \\ 0 & 1 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & 1 & a_{n-1} \end{pmatrix} \qquad (*)$$

Furthermore, the characteristic polynomial of t is

$$\begin{aligned} P_t(X) &= \det(X.I_n - C(t)) \\ &= X^n - a_{n-1}X^{n-1} - a_{n-2}X^{n-2} - \ldots - a_0 , \end{aligned}$$

so $a_0, a_1, \ldots, a_{n-1}$ are invariants of the conjugacy class in G of t; they determine the class by $(*)$

<u>Corollary 1.57</u>

The number of regular conjugacy classes in G is $(q-1)q^{nh-1}$.

<u>Proof</u>:

The definition of regularity is invariant under conjugacy, so the statement makes sense; in G the elements are invertible so $\det(t) \in R^*$, i.e. $(-1)^{n-1}a_0 \in R^*$, so there are $(q-1)q^{h-1}$ choices for a_0. The remaining a_i may be chosen arbitrarily and there are q^h choices for each. ∎

<u>Proposition 1.58</u>

(i) $t \in \mathrm{End}(V)$ is regular iff $\bar{t}$ is regular

(ii) $t \in GL(V)$ is regular iff $Z_{GL(V)}(t)$ is abelian.

<u>Proof</u>: These results are due to G. Lusztig except the implication $\Leftarrow$ of (ii) which follows from (i) and the corresponding result over the finite field $\bar{R}$ using the Teichmuller section. ∎

<u>Proposition 1.59</u>

Let $t \in G$ be unramified semisimple with eigenvalues $t_1, \ldots, t_n \in R'^*$ and let $\{e_1, \ldots, e_n\}$ be a basis of V' with $t(e_i) = t_i e_i$

for each i. Then t is regular iff $k_\alpha(t) = 0$ for all $\alpha \in \Phi$ iff $(t_i - t_j) \in R'^*$ for $1 \leq i, j \leq n$, $i \neq j$.

Proof:

t is regular iff there exist $a_1, \ldots, a_n \in R'$ such that $\{ \sum_{i=1}^{n} t_i^j a_i e_i : 0 \leq j \leq n-1 \}$ forms a basis of V' iff there exist $a_1, \ldots, a_n \in R'$ such that

$$\det \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ t_1 a_1 & t_2 a_2 & \cdots & t_n a_n \\ \vdots & \vdots & & \vdots \\ t_1^{n-1} a_1 & \cdots & \cdots & t_n^{n-1} a_n \end{pmatrix} \in R'^*$$

iff there exist $a_1, \ldots, a_n \in R'$ such that

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & \vdots & & \vdots \\ t_1^{n-1} & \cdots & \cdots & t_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \in R'^*$$

iff $\det \mathcal{V}(t) \in R'^*$ (taking $a_1 = a_2 = \ldots = a_n = 1$),

where $\mathcal{V}(t)$ is the Van der Monde matrix just above :

$\det \mathcal{V}(t) = (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (t_i - t_j)$ and this lies in R'^*

iff $(t_i - t_j) \in R'^*$ for $1 \leq i < j \leq n$

iff $k_\alpha(t) = 0$ for all $\alpha \in \Phi$. (By §1.54(iii)). ∎

Finally, we show that there is a unique regular unipotent conjugacy class in G (a unipotent element is one conjugate to an element of U); this result is well-known if R is a field.

Proposition 1.510

The element $\begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 1 & a_{23} & \cdots & \vdots \\ & & 1 & \ddots & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{pmatrix} \in U$ is regular iff $a_{i,i+1} \in R^*$ $(1 \leq i \leq n-1)$, in which case it is conjugate to

$$u_0 = \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}.$$

<u>Proof</u>:

$u \in U$ is regular iff there exist $a_1,\dots,a_n \in R$ such that $\{u^j \sum_{i=1}^{n} a_i e_i \,:\, 0 \leq j \leq n-1\}$ forms a basis of V iff $\det(M) \in R^*$ where M is the matrix with i'th column $u^{i-1}\underline{a}$ $(1 \leq i \leq n)$ and

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \text{ Write } m_i = a_n . \prod_{j=i}^{n-1} a_{j,j+1} \quad (1 \leq i \leq n-1) \text{ and } m_n = a_n .$$

The following sequence of column operations brings M to triangular form $\begin{pmatrix} & & & m_1 \\ & * & m_2 & \\ & \cdot & & \\ m_n & & & 0 \end{pmatrix}$ but leaves its determinant unchanged :

$$K_n' = K_n - K_{n-1},\ K_{n-1}' = K_{n-1} - K_{n-2},\ \dots,\ K_2' = K_2 - K_1$$

$$K_n'' = K_n' - K_{n-1}',\ \dots\dots,\ K_3'' = K_3' - K_2'$$

$$\dots\dots$$

$$K_n^{(n-1)} = K_n^{(n-2)} - K_{n-1}^{(n-2)} .$$

Hence $\det(M) = (-1)^{[\frac{n}{2}]} \prod_{i=1}^{n-1} a_{i,i+1}^{i} . a_n^n$ whence the condition $a_{i,i+1} \in R^*$ $(1 \leq i \leq n-1)$ is clear.

If this condition holds choose $a_1 = 1$, $a_i = \prod_{j=1}^{i-1} a_{j,j+1}$ $(2 \leq i \leq n)$. Then

$$tut^{-1} = \begin{pmatrix} 1 & 1 & b_{13} & \cdot & \cdot & b_{1n} \\ & 1 & 1 & \cdot & \cdot & \cdot \\ & & 1 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & 1 \\ & 0 & & & & 1 \end{pmatrix} \text{ where } t = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix},$$

and we may reduce this further to the form u_0 by conjugating by $\prod_{2 \leq i < j \leq n} x_{ij}(b_{ij})$, the order of the product being given by : $x_{ij}(b_{ij})$ appears to the left of $x_{lm}(b_{lm})$ iff : $i \geq l$ and if $i = l$ then $j \geq m$. ■

Note : the above proof holds over any (commutative) ring.

Chapter 2

The representation S_G

We shall now construct a representation S_G of G which coincides with the usual Steinberg representation ([17]) in case R is a field and which, like the latter, has 'nice' character values : we show (in §2.2) that its character is $\pm$ a power of q at unramified semisimple elements and zero at certain non-trivial unipotent elements.

S_G will be defined as an alternating sum of representations induced from parabolic subgroups, comparable to Curtis' formula ([5]), but S_G is not irreducible if $h \geq 2$ nor, it seems, constructible homologically, although it can be written as an alternating sum of permutation representations (§2.14). In §2.3 we study the restriction of S_G to a parabolic subgroup of G showing, in particular, that at split semisimple elements S_G behaves as if it were the permutation representation of G on the set of minimal parabolic subgroups 'opposite' to a fixed minimal parabolic subgroup.

2.1 The definition of S_G

The maps $\pi_i : R \to R_i = R/\mathfrak{m}^i$ $(0<i<h)$ of §1.0 induce maps $\pi_i : V \to V/\pi^i V$, $\pi_i : G \to G(i) = GL(V/\pi^i V)$ and $\pi_i : H \to H(i)$ for any subgroup H of G. Denote the kernel of this last map by $H(\pi^i R)$. All the structure theorems proved for G in chapter 1 apply equally to G(i) if we replace h by i.

We define S_G inductively. Assume that S_M is already defined for groups M of the form $M = GL(W)$, W a free module over R_{h-1}; we start with the case h=1, where M = 1, so $S_M = 1$. Then if $L = M_1 \times \ldots \times M_r$ with $M_i = GL(W_i)$, W_i a free R_{h-1}-module, we define $S_L = S_{M_1} \otimes S_{M_2} \otimes \ldots \otimes S_{M_r}$. -----(2.11).

Thus $S_{L_X(h-1)}$ is defined for each $X \subset S$ (see §1.24) and we may regard it as a representation of P_X by the maps $P_X \to L_X \to L_X(h-1) \to \mathrm{Aut}(S_{L_X(h-1)})$ (the first map comes from §1.27). Now induce this representation of P_X to G and denote it by $I_X = (S_{L_X(h-1)})^G_{P_X}$. Of course I_X is independent of the particular choice of parabolic and Levi subgroups of type X : $I_X = (S_{L(h-1)})^G_P$ for any P conjugate to P_X with Levi component L (see §1.13 and §1.27).

<u>Definition 2.12</u>

$$S_G = \sum_{X\subset S} (-1)^{|X'|} I_X .$$

<u>Remarks</u>

(i) S_G coincides with the usual Steinberg representation in case R is a field (h=1) since $S_{L_X(0)}=1$ for any $X \subset S$, and we have $I_X = 1^G_{P_X}$ so §2.12 reduces to Curtis' formula ([5],Thm.2).
(ii) If $M = GL(W)$ with W free of rank 1 over R then $S_M = I_S = S_{M(h-1)} = \dots = S_{M(0)} = 1$. So by §1.24 and §2.11 we see $S_{L_S(h-1)} = 1$ and hence $I_S = 1^G_B$. In chapter 4 we shall show that S_G is a subrepresentation of 1^G_B, (except possibly if q=2).
(iii) In view of our determination in §1.2 of the parabolic subgroups of L and their Levi components, §2.11 shows that, just as for S_G in §2.12, S_L is an alternating sum of representations induced from parabolic subgroups of L.

We shall now show how to express S_G as an alternating sum of permutation representations.

Given a sequence $\mathbf{k} = (k_1,\dots,k_{n-1})$ of integers with $0\le k_i \le h$ $(1\le i\le n-1)$ define $k_{ij} = \max\{k_r : j\le r<i\}$ for $1\le j<i\le n$.

Write $G = GL_n(R)$ (with respect to the basis $\underset{\sim}{e}$ of V) and define $H_{\mathbf{k}} = \{\text{matrices } (m_{ij}) \in G : \pi^{k_{ij}} | m_{ij} \text{ for } 1\le j<i\le n\}$.

Lemma 2.13

$H_{\mathbf{k}}$ is a subgroup of G.

Proof:

Suppose $M = (m_{ij})$, $M' = (m'_{ij}) \in H_{\mathbf{k}}$. If $M.M' = (n_{ij})$ then $n_{ij} = \sum_{k=1}^{n} m_{ik}m'_{kj}$. Suppose $i > j$.
For $k < i$ let $a_k = \max\{k_r : k \leq r < i\}$; so $\pi^{a_k} | m_{ik}$. For $k > j$ let $b_k = \max\{k_r : j \leq r < k\}$; so $\pi^{b_k} | m'_{kj}$. Let $m = \max\{k_r : j \leq r < i\}$. Then for $k \leq j$ we have $m \leq a_k$ and for $i \leq k$ we have $m \leq b_k$, so $\pi^m | m_{ik}m'_{kj}$ for $k \leq j$ and for $i \leq k$. If $j<k<i$ then either $m=a_k$ or $m=b_k$ so $\pi^m | m_{ik}m'_{kj}$ in this case too. So $\pi^m | n_{ij}$ and thus $H_{\mathbf{k}}$ is closed under multiplication.

Hence $H_{\mathbf{k}}$ is generated (as a multiplicative set) by the subgroups B and $x_{ij}(\pi^{k_{ij}}R)$ $(1 \leq j<i \leq n)$ of G, so is a subgroup of G. ∎

Remark

The groups $H_{\mathbf{k}}$ are precisely those which occur in constructing the principal series of G ([10]); see §3.6.

We now associate to $\mathbf{k}$ a filtration $\mathcal{S}=(\emptyset=S_{h+1} \subset S_h \subset .. \subset S_1 \subset S_0=S)$ of S given by $S_i = \{r \in S : i \leq k_r\}$ $(1 \leq i \leq h)$.

Then we have a bijection for each k with $0 \leq k \leq h$:

$$\beta_k : \{\mathbf{k} \text{ with all } k_r \leq k \text{ and at least one } k_r \text{ equal to } k\} \rightarrow \{\mathcal{S} \text{ with } S_k \neq \emptyset, \text{ and } S_{k+1} = \ldots = S_{h+1} = \emptyset\}$$

The inverse is given by $\beta_k^{-1}(\mathcal{S}) = (k_1,\ldots,k_{n-1})$ with $k_r = \max\{j : r \in S_j\}$ if such exists, and 0 otherwise.

Together the β_k give a bijection

$$\beta : \{\text{all sequences } \mathbf{k}\} \rightarrow \{\text{all filtrations } \mathcal{S}\}.$$

Define $|\mathbf{k}| = \sum_{r=1}^{n-1}(h-k_r)$ and $|\mathcal{S}| = \sum_{i=1}^{h}|S'_i|$. Then if $\mathcal{S} = \beta(\mathbf{k})$ we have $|\mathbf{k}| = |\mathcal{S}|$; in fact $(h-k_1,\ldots,h-k_{n-1})$ (suitably reordered) and $(|S'_h|,\ldots,|S'_1|)$ are dual partitions.

We shall often write $H_{\mathcal{S}}$ for $H_{\mathbf{k}}$ when $\mathcal{S} = \beta(\mathbf{k})$.

<u>Proposition 2.14</u>

$$S_G \cong \sum_{\mathfrak{k}} (-1)^{|\mathfrak{k}|} 1_{H_{\mathfrak{k}}}^G \quad \text{(sum over all sequences } \mathfrak{k}\text{)}.$$

<u>Proof</u>:

We use induction on h. So assume that the corresponding result is true for all groups of the form $GL_m(R_{h-1})$; this is trivial for h=1, which starts the induction, since then the only $\mathfrak{k}$ is $(0,\dots,0)$ and $H_{\mathfrak{k}} = GL_m(R_0)$, $S_{H_{\mathfrak{k}}} = 1$.

Now if $X = \{i_1,\dots,i_r\}$ with $i_1<\dots<i_r$ we have

$L_X(h-1) = \{M = (m_{ij}) \in GL_n(R_{h-1}) : m_{ij}=0 \text{ if } \alpha_{ij} \in \Phi - \Phi_{X'}\}$

$\cong G_1 \times \dots \times G_{r+1}$, where for $1 \leq s \leq r+1$

$G_s = \{M = (m_{ij}) \in GL_n(R_{h-1}) : m_{ii}=1 \text{ for } 1 \leq i \leq i_{s-1}, i_s < i \leq n, \text{ and}$
$\text{if } i<j \text{ then } m_{ij}=0=m_{ji} \text{ except possibly for } i_{s-1}<i<j\leq i_s\}$.

Write $X_s = \{i : i_{s-1}<i<i_s\}$. By inductive hypothesis, $S_{G_s} = \sum_{\tau} (-1)^{|\tau|} 1_{H_\tau}^{G_s}$, sum over all filtrations $\tau = (T_{h-1} \subset \dots \subset T_1)$ of X_s. H_τ denoting the corresponding subgroup of G_s.

As a representation of $L_X(h-1)$ $S_{G_s} = \sum_{\tau} (-1)^{|\tau|} 1_{H'_\tau}^{L_X(h-1)}$, where H'_τ is the subgroup of $L_X(h-1)$ corresponding to the filtration $(T_{h-1} \subset \dots \subset T_1)$ of X'. Thus

$$H'_\tau \cong G_1 \times \dots \times G_{s-1} \times H_\tau \times G_{s+1} \times \dots \times G_{r+1}.$$

Now if $\tau_s = (T^s_{h-1} \subset \dots \subset T^s_1)$ is a filtration of X_s $(1 \leq s \leq r+1)$ then $H_{\tau_s} \cap H_{\tau_t} = 1$ for $1 \leq s < t \leq r+1$ and $H_{\tau_1} \times \dots \times H_{\tau_{r+1}} \cong H_{\mathfrak{R}}$, the subgroup of $L_X(h-1)$ defined by the filtration $\mathfrak{R} = (R_{h-1} \subset \dots \subset R_1)$ of X', where $R_i = \bigcup_{s=1}^{r+1} T_i^s$ (disjoint union) $(1 \leq i \leq h-1)$.

So noting that $|\mathfrak{R}| = \sum_{1 \leq s \leq r+1} |\tau_s|$ we have

$$\bigotimes_{s=1}^{r+1} (-1)^{|\tau_s|} 1_{H'_{\tau_s}}^{L_X(h-1)} \cong (-1)^{|\mathfrak{R}|} 1_{H_{\mathfrak{R}}}^{L_X(h-1)}.$$

Hence $S_{L_X(h-1)} = S_{G_1} \times \dots \times S_{G_{r+1}}$

$$\cong \sum_{\mathfrak{R}} (-1)^{|\mathfrak{R}|} 1_{H_{\mathfrak{R}}}^{L_X(h-1)},$$

the sum being over all filtrations $\mathfrak{R} = (R_{h-1} \subset .. \subset R_1)$ of X', and as a representation of P_X $S_{L_X(h-1)} \cong \sum_{\mathcal{S}} (-1)^{|\mathcal{S}|-|X'|} 1_{H_{\mathcal{S}}}^{P_X}$, sum over all filtrations $\mathcal{S} = (S_h \subset .. \subset S_1)$ of S with $S_h = X$.

(Note that $S_h = X$ implies $H_{\mathcal{S}} \subset P_X$; and $(H_{\mathcal{S}} \cap L_X)(h-1) = H_{\mathfrak{R}}$ where $\mathfrak{R} = (S_{h-1}-X \subset \ldots \subset S_1 - X)$).

Hence $I_X \cong (-1)^{|X'|} \Sigma (-1)^{|\mathcal{S}|} 1_{H_{\mathcal{S}}}^G$, sum over all $\mathcal{S}$ with $S_h = X$, and so $S_G \cong \sum_{\mathcal{S}} (-1)^{|\mathcal{S}|} 1_{H_{\mathcal{S}}}^G$. ∎

2.2 The character theorems

Let $t \in G$ be unramified semisimple and assume that L_X is minimal for the property of being a Levi subgroup of G containing t. For each i with $1 \leq i \leq h$ there exists a unique maximal (for inclusion) subset $X(i)$ of S such that $\pi_i(t)$ lies in $L_{X(i)}(i)$; note that $X(h) = X$. Then $X(i) \subset X(j)$ if $j < i$ (since for example $\pi_i(L_Y) = L_Y(i)$) and $|X(i)|$ is the number of orbits of F on the set of eigenvalues of $\pi_i(t)$ (see §1.5).

Theorem 2.21

Let t be unramified semisimple and $X(i)$ as above. Then the character of S_G on t is given by $S_G(t) = (-1)^x . q^k$, where $x = \sum_{i=1}^{h} |X(i)'|$ and $k = \sum_{\alpha \in \Phi^+} k_\alpha(t)$.

Remark

Suppose that $L_X = \prod_{i=1}^{r} GL(M_i^F)$, as in §1.5, and assume that t acts regularly on each M_i^F, i.e. that $k_\alpha(t) = 0$ for all $\alpha \in \Phi_{X'}$ (see §1.59). This assumption certainly holds if $h=1$ since $k_\alpha(t) \neq 0$ implies $k_\alpha(t) = 1$ and so, for example, $\xi = F^i \xi$, a contradiction. Then

$$
\begin{aligned}
|Z_G(t) \cap U_X| &= |\{u \in U_X : ut = tu\}| \\
&= q^{\sum_{\alpha \in \Phi^+ - \Phi^+_{X'}} k_\alpha(t)} \qquad \text{(see §2.24)}
\end{aligned}
$$

$= q^k$ since $k_\alpha(t) = 0$ for $\alpha \in \Phi^+_{X'}$.

Hence $S_G(t) = (-1)^x |Z_G(t) \cap U_X|$. In case R is a field (h=1) this equals $(-1)^{|X'|} . |Z_G(t)|_p$, which is well-known ([17],§5.7).

Theorem 2.22

Suppose $t \in P_X = L_X.U_X$ has (unique) form $t = l.m$ with $l \in L_X$ unramified semisimple, L_X minimal for being a Levi subgroup of G containing l, and $m \in U_X$. Assume that $\pi_{h-1}(l) = 1 = \pi_{h-1}(m)$, but $m \neq 1$.

Then $S_G(t) = 0$.

Remark

In case R is a field it follows from the Jordan canonical form for elements of G that theorems 2.21/2 completely compute the character of S_G. In particular the character vanishes except at semisimple elements, where it is $\pm$ a power of q. (This is also given in [17],§5.7). At regular semisimple elements it is ± 1; correspondingly, if $h \geq 1$, $S_G(t) = \pm 1$ if t is regular unramified semisimple (§1.59,§2.21).

The dimension of S_G is q^{Nh}, where $N = |\Phi^+| = n(n-1)/2$.

For convenience, and to emphasise the similarities between them, the proofs of theorems 2.21/2 will run concurrently; we shall refer to the situation of §2.21 as 'Case 1' and to that of §2.22 as 'Case 2', and to unify notation we shall write t=l and m=1 in Case 1. The first step is to establish :

Proposition 2.23

Let $Y \subset S$. Then

$I_Y(t) = \Sigma\, S_{L_Y(h-1)}(w^{-1}lw) . |\{\sigma \in \mathcal{F}_Y(w)$ fixed by $t\}|$, the sum being taken over those $w \in \underline{W}_Y$, such that $w^{-1}lw \in L_Y$.

<u>Proof</u>:

$\mathcal{Y} = \{g = uw : w \in \underline{W}_{Y'},\ u = \prod x_\alpha(r_\alpha) \in {}^{w}U_Y^- \text{ with } \pi | r_\alpha \text{ if } \alpha<0\}$ is the set of distinguished representatives of P_Y in G (§1.46). So $I_Y(t) = \Sigma\, S_{L_Y(h-1)}(g^{-1}tg)$, the sum being taken over the set $\{g \in \mathcal{Y} : g^{-1}tg \in P_Y\}$ or, what amounts to the same thing, the set $\{g \in \mathcal{Y} : t \text{ fixes } \sigma = g.\rho_Y\}$.

Let $g = uw \in \mathcal{Y}$ and write $l_1 = w^{-1}lw$, $u_1 = w^{-1}u^{-1}w$; note that $u_1 \in U_Y^-$. Then $g^{-1}lg = u_1 l_1 u_1^{-1}$. Also we may write, uniquely, $u_1 = u_2u_3$ with $u_2 \in U_Y^- \cap \prod_{\alpha\in\Phi^- - (w^{-1}\Phi_{X'})^-} U_\alpha$ and $u_3 \in U_Y^- \cap \prod_{\alpha\in(w^{-1}\Phi_{X'})^-} U_\alpha = U_Y^- \cap w^{-1}L_Xw$; this last equality is because $w^{-1}L_Xw = \langle T, U_\alpha : \alpha \in w^{-1}\Phi_{X'} \rangle$, which follows from §1.24. §1.46 and the facts $w^{-1}Tw = T$ and (§1.37) $w^{-1}U_\alpha w = U_{w^{-1}\alpha}$.

Now put $l_2 = u_3l_1u_3^{-1} \in w^{-1}L_Xw$ and note that L_X is minimal for being a Levi subgroup of G containing wl_2w^{-1}. Then

$u_1l_1u_1^{-1} = u_2l_2u_2^{-1} = l_2.[l_2,u_2^{-1}]$, and

$$[l_2,u_2^{-1}] = (l_2^{-1}u_2l_2).u_2^{-1} \in U_Y^- \cap \prod_{\alpha\in\Phi^- -(w^{-1}\Phi_{X'})^-} U_\alpha = \prod_{\alpha\in\Phi^- - \Phi_{Y'}^- - (w^{-1}\Phi_{X'})^-} U_\alpha .$$

Also $u = \prod x_\alpha(r_\alpha) \in {}^{w}U_Y^-$ with $\pi|r_\alpha$ if $\alpha<0$ and $m = \prod x_\alpha(s_\alpha) \in U_X$ with $\pi^{h-1}|s_\alpha$ for all α, so by §1.34 $u^{-1}mu = v^{-1}mv$ where $v = \prod_{\alpha\in\Phi^+\cap w(\Phi^- - \Phi_{Y'}^-)} x_\alpha(r'_\alpha) \in U \subset P_X$, so since $U_X \triangleleft P_X$ we have $u^{-1}mu \in U_X$. Hence $g^{-1}mg = w^{-1}u^{-1}muw \in {}^{w^{-1}}U_X$, and $g^{-1}mg = v_1v_2$ with $v_2 \in P_Y$ and $v_1 \in U_Y^- \cap {}^{w^{-1}}U_X$, which is contained in $U_Y^- \cap \prod_{\alpha\in\Phi^- -(w^{-1}\Phi_{X'})^-} U_\alpha$.

So $g^{-1}tg \in P_Y$ iff $l_2.[l_2,u_2^{-1}].v_1v_2 \in P_Y$ iff there exists $p \in P_Y$ with $[l_2,u_2^{-1}]v_1 = l_2^{-1}pv_2^{-1}$.

But $l_2^{-1}pv_2^{-1} \in (w^{-1}L_Xw).P_Y \subset \langle T, U_\alpha : \alpha \in \Phi^+ \cup \Phi_{Y'}^- \cup (w^{-1}\Phi_{X'})^- \rangle$ and $[l_2,u_2^{-1}]v_1 \in \prod_{\alpha\in\Phi^- - \Phi_{Y'}^- - (w^{-1}\Phi_{X'})^-} U_\alpha$:

so $g^{-1}tg \in P_Y$ iff $l_2^{-1}pv_2^{-1} = 1 = [l_2, u_2^{-1}]v_1$.
Now $l_2 = pv_2^{-1} \in P_Y$ implies $w^{-1}L_Xw \subset L_Y$ by §1.53(i), whence $w^{-1}L_Xw \cap U_Y^- = 1$ and so $u_3=1$, $l_2=l_1$, $u_1=u_2$; and we have $w^{-1}lw \in L_Y$. Finally :

<u>Case 1</u> : $1 = [l_2, u_2^{-1}] = [l_1, u_1^{-1}] = w^{-1}[l,u]w$ implies that $[l,u] = 1$ and so $g^{-1}tg = w^{-1}lw$.
Thus if $\sigma = g.\rho_Y$ is fixed by t, $S_{L_Y(h-1)}(g^{-1}tg) = S_{L_Y(h-1)}(w^{-1}lw)$ and §2.23 follows in this case.

<u>Case 2</u> : Since $\pi_{h-1}(t) = 1 = \pi_{h-1}(l)$ we have $\pi_{h-1}(g^{-1}tg) = 1 = \pi_{h-1}(w^{-1}lw)$ also, and so $S_{L_Y(h-1)}(g^{-1}tg) = \dim(S_{L_Y(h-1)}) = S_{L_Y(h-1)}(w^{-1}lw)$. ∎

The foregoing proof also leads to the following consequences which will be required later.

<u>Corollary 2.24</u>

(i) Let $\sigma = g.\rho_Y$ with $g = uw \in \mathcal{Y}$, $u = \pi x_\alpha(r_\alpha)$ (product over $\alpha \in w(\Phi^- - \Phi^-_{Y'})$). Let t be Case 1, assume that $l_1 = w^{-1}lw \in L_Y$, and write $u_1 = w^{-1}u^{-1}w$. Then t fixes σ iff $[l_1, u_1^{-1}] = 1$ iff $\alpha({}^cl_1).r_{w\alpha} = r_{w\alpha}$ for all $\alpha \in \Phi^- - \Phi^-_{Y'}$, for a suitable c.

In particular, if $Y = w_0(X)$ and $w = w_0w_{Y'}$ then

$$|\{\sigma \in \mathcal{F}_Y(w) \text{ fixed by } t\}| = q^{\sum_{\alpha \in \Phi^+ - \Phi^+_{X'}} k_\alpha(t)} .$$

(ii) Let t be Case 2 and $Y = w_0(X)$, $w = w_0w_{Y'}$. Then

$$|\{\sigma \in \mathcal{F}_Y(w) \text{ fixed by } t\}| = 0.$$

<u>**Proof**</u>:

(i)Suppose l_1 splits over R', so there exists $c \in (L_Y)'$ (denoting by a ' the group over R') such that ${}^cl_1$ stabilises each free rank 1 R'-module $R'.e_i$ (i.e. ${}^cl_1 \in (T)'$). Now by §1.46 we may write c in the form $c = w_ctv_1v_2$ with $t \in (T)'$,

$w_c \in W_{Y'}$, $v_1 \in (U^- \cap L_Y)'$ and $v_2 \in (U \cap L_Y)'$; but if ${}^c l_1 \in (T)'$ then $w_c^{-1}.cl_1c^{-1}.w_c \in w_c^{-1}.(T)'.w_c = (T)'$, so we may assume $w_c = 1$.

But by the proof of §2.23, t fixes σ iff $[l_1, u_1^{-1}] = 1$, which, by §1.56(iii) is equivalent to $\alpha({}^c l_1).r_{w\alpha} = r_{w\alpha}$ for all $\alpha \in \Phi^- - \Phi^-_{Y'}$.

Now $w_{Y'} \in W_{Y'}$ so permutes the elements of $\Phi^- - \Phi^-_{Y'}$ amongst themselves and so fixes U_Y^-. So if $w = w_0 w_{Y'}$ and $Y = w_0(X)$ then ${}^w U_Y^- = {}^{w_0} U^-_{w_0(X)} = U_X$ by §1.37. Note that $w \in \underline{W}_{Y'}$ and $l_1 = w^{-1} l w \in L_Y$ (§1.53(ii)). Thus $|\{\sigma \in \mathcal{F}_Y(w) \text{ fixed by } t\}|$
$= |\{u = \prod x_\alpha(r_\alpha) \in {}^w U_Y^- : \alpha({}^c l_1) r_{w\alpha} = r_{w\alpha} \text{ for all } \alpha \in \Phi^- - \Phi^-_{Y'}\}|$

$$= q^{\sum_{\alpha \in \Phi^- - \Phi^-_{Y'}} k_\alpha({}^c l_1)} = q^{\sum_{\alpha \in \Phi^- - \Phi^-_{Y'}} k_\alpha(l_1)} \quad \text{since } c \text{ lies over } w=1,$$

$$= q^{\sum_{\alpha \in w(\Phi^- - \Phi^-_{Y'})} k_\alpha(wl_1w^{-1})} = q^{\sum_{\alpha \in \Phi^+ - \Phi^+_{X'}} k_\alpha(t)} .$$

(<u>Remark</u>: since ${}^w U_Y^- = U_X$ we see $|\{u \in U_X : ut = tu\}| = q^{\sum_{\alpha \in \Phi^+ - \Phi^+_{X'}}}$).

(ii) From the proof of §2.23, t fixes $\sigma = uw.\rho_{w_0(X)}$ $(u \in U_X)$ iff $w^{-1}u^{-1}tuw \in P_{w_0(X)}$ iff $u^{-1}tu \in P_X^!$, since ${}^w P_{w_0(X)} = {}^w L_{w_0(X)} . {}^w U_{w_0(X)} = L_X . U_X^- = P_X^!$.
But $u \in U_X \subset P_X$ and $t \in P_X$ so $u^{-1}tu \in P_X$ and thus t fixes σ iff $u^{-1}tu \in L_X = P_X^! \cap P_X$. Also $t = lm$ with $l \in L_X$, $m \in U_X$ so $l^{-1}.u^{-1}tu = l^{-1}u^{-1}l.m.u \in U_X$. But if $u^{-1}tu \in L_X$ then $l^{-1}.u^{-1}tu \in L_X$, so since $U_X \cap L_X = 1$, $m = l^{-1}.ulu^{-1}$ and this lies in $L_X . {}^u L_X$ which intersects U_X in 1. So $m=1$, which contradicts the hypothesis of §2.22. So t cannot fix σ. ∎

To simplify notation we put $w_1 = w_0 w_{w_0(X)}$, and if $Y \subset S$, $w \in \underline{W}_{Y'}$ put $T_Y(w) = S_{L_Y(h-1)}(w^{-1}lw).|\{\sigma \in \mathcal{F}_Y(w) \text{ fixed by } t\}|$.

Now $I_Y(t) = 0$ except, perhaps, when P_Y contains a conjugate $g^{-1}tg$ of t, where we may assume $g = uw \in \mathcal{Y}$, in which case, by the proof of §2.23, we may also assume that $w^{-1}lw \in L_Y$, so in fact $w^{-1}L_Xw \subset L_Y$ by §1.53(i). Then using §1.53(ii) we have $S_G(t) = (-1)^{|w_0(X)'|}T_{w_0(X)}(w_1) + \mathcal{R}$. where the remainder $\mathcal{R}$ is a sum of terms each of the form $\sum_{Z\subset U\subset Y} (-1)^{|U|}T_U(w)$ ---(*), with $w \in C_Z$, L_Y conjugate to L_X, and $Z \neq Y$: note that if $w \in C_Z$ then $T_U(w)$ doesn't occur except possibly for $Z \subset U$, since we must have $w \in \underline{W}_{U'}$.

The second stage of the proof of §2.21/2 is to show

Proposition 2.25

Let $w \in C_Z$ and $Z \subset U \subset Y$. Then $T_U(w) = T_Y(w)$.

It follows that each term like (*) is 0 (since $Z \neq Y$), and so $\mathcal{R} = 0$ and we have

(2.26) $\quad S_G(t) = (-1)^{|w_0(X)'|}T_{w_0(X)}(w_1)$.

The proof of §2.25 depends on

Proposition 2.27

Let $U \subset Y \subset S$, $w \in \underline{W}_{U'}$, and suppose $l_1 = w^{-1}lw \in L_Y$. Write $k'_\alpha = k_\alpha(\pi_{h-1}({}^cl_1))$ and $\lambda = \sum_{\alpha\in\Phi^+_{U'},-\Phi^+_{Y'}} k'_\alpha$ where c is as in §2.24(i). Then

$|\{\sigma\in\mathcal{F}_Y(w) \text{ fixed by } t\}| = q^\lambda.|\{\sigma\in\mathcal{F}_U(w) \text{ fixed by } t\}|$.

Assuming §2.27 for the moment we complete the proof of §2.25.

Case 1 : We require an inductive hypothesis, namely that §2.21 holds for groups of the form $GL(W)$, W a free R_{h-1}-module; then it holds also for the groups $L_Y(h-1)$ (§1.24). The induction starts trivially with the case h=1.

Then $S_{L_Y(h-1)}(w^{-1}tw).q^{\lambda} = (-1)^y.q^{\sum_{\alpha\in\Phi^+_{Y'}} k_\alpha(\pi_{h-1}(l_1))}.q^{\lambda}$

$= (-1)^y.q^{\sum_{\alpha\in\Phi^+_{Y'}} k'_\alpha}.q^{\sum_{\alpha\in\Phi^+_{U'}-\Phi^+_{Y'}} k'_\alpha}$, using §1.55,

$= (-1)^y.q^{\sum_{\alpha\in\Phi^+_{U'}} k'_\alpha} = S_{L_U(h-1)}(w^{-1}lw)$ ----(*);

the sign $(-1)^y$ is the same in each case since if $w^{-1}lw \in L_Y$ then $\pi_{h-1}(w^{-1}lw) \in L_Y(h-1) \subset L_U(h-1)$, so the sequences of minimal parabolic subgroups of $L_Y(i)$, $L_U(i)$ respectively, for $1\leq i\leq h-1$, which determine the signs of $S_{L_Y(h-1)}(w^{-1}lw)$, $S_{L_U(h-1)}(w^{-1}lw)$ respectively, are the same.

But $T_Y(w) = S_{L_Y(h-1)}(w^{-1}lw).q^{\lambda}.|\{\sigma\in\mathcal{F}_U(w) \text{ fixed by } t\}|$

$= T_U(w)$ by (*).

<u>Case 2</u>

$S_{L_Y(h-1)}(w^{-1}lw) = \dim(S_{L_Y(h-1)}) = q^{|\Phi^+_{Y'}|.(h-1)}$ and since $\pi_{h-1}(w^{-1}lw) = 1$, $k'_\alpha = h-1$ for all α, so $\lambda = |\Phi^+_{U'}-\Phi^+_{Y'}|.(h-1)$.

So $S_{L_Y(h-1)}(w^{-1}lw).q^{\lambda} = q^{|\Phi^+_{U'}|.(h-1)}$

$= \dim(S_{L_U(h-1)})$

$= S_{L_U(h-1)}(w^{-1}lw)$,

so (*) also holds in this case, and $T_Y(w) = T_U(w)$ as before. ∎

<u>Proof of §2.27</u>

For each $v \in U(w) = \prod_{\alpha\in\Phi^-\cap w(\Phi^-_{U'}-\Phi^-_{Y'})} U_\alpha(\pi R)$ we define a map $\theta_v:\mathcal{F}_U(w) \to \mathcal{F}_Y(w)$ as follows.

Suppose $v = \prod x_\alpha(r_\alpha)$ where $r_\alpha \in \pi R$ (we assume chosen a fixed order for all such products), and suppose $\sigma\in\mathcal{F}_U(w)$ has distinguished representative uw with $u = \prod_{\alpha\in w(\Phi^--\Phi^-_{U'})} x_\alpha(r_\alpha)$. Put $g = \prod_{\alpha\in w(\Phi^--\Phi^-_{Y'})} x_\alpha(r_\alpha).w$ and define

$$\theta_v(\sigma) = g.\rho_Y \quad \text{----(**)}.$$

Now $\Phi^--\Phi^-_{Y'} = (\Phi^--\Phi^-_{U'})\cup(\Phi^-_{U'}-\Phi^-_{Y'})$ and $\underline{W}_{U'} \subset \underline{W}_{Y'}$ so g is the distinguished representative for $\theta_v(\sigma)$.

In particular, θ_v is injective.

Further, the union $\bigcup_{v\in U(w)} \theta_v(\mathcal{F}_U(w))$ is disjoint since if $v_1, v_2 \in U(w)$ and $v_1 \neq v_2$ then the distinguished representatives g_1, g_2 of $\theta_{v_1}(\sigma) = g_1 \cdot \rho_Y$, $\theta_{v_2}(\sigma) = g_2 \cdot \rho_Y$ will be different (by §1.32 : uniqueness of expression of products $\prod x_\alpha(r_\alpha)$).

We now observe that $w \in \underline{W}_{U'}$ implies $w\alpha < 0$ for $\alpha \in \Phi^-_{U'} - \Phi^-_{Y'}$, whence $\Phi^+ \cap w(\Phi^- - \Phi^-_{Y'}) = \Phi^+ \cap w(\Phi^- - \Phi^-_{U'})$ (and, incidentally, $U^- \cap w^{-1}Uw = U^-_U \cap w^{-1}Uw$), so we obtain all distinguished representatives g for elements of $\mathcal{F}_Y(w)$ by the process of (**). Hence $\mathcal{F}_Y(w) = \bigcup_{v\in U(w)} \theta_v(\mathcal{F}_U(w))$, so we reduce to proving

Lemma 2.28

With the assumptions and notations of §2.27, let $\sigma \in \mathcal{F}_U(w)$. Then t fixes σ iff t fixes $\theta_v(\sigma)$ for all $v = \prod x_\alpha(r_\alpha)$ in $U(w)$ such that $r_{W\alpha}\pi^{k'_\alpha} = 0$ $(\alpha \in \Phi^-_{U'} - \Phi^-_{Y'})$.

Proof:

Case 1 :

Let $\sigma = uw \cdot \rho_U$, $u = \prod x_\alpha(r_\alpha) \in {}^W U^-_U$. Since $l_1 = w^{-1} l w \in L_Y \subset L_U$, by §2.24(i) t fixes σ iff $\alpha({}^c l_1) r_{W\alpha} = r_{W\alpha}$ for all $\alpha \in \Phi^- - \Phi^-_{U'}$, and similarly if $v = \prod x_\alpha(r_\alpha) \in U(w)$ then t fixes $\theta_v(\sigma)$ iff $\alpha({}^c l_1) r_{W\alpha} = r_{W\alpha}$ for all $\alpha \in \Phi^- - \Phi^-_{Y'}$. Hence t fixes σ iff t fixes $\theta_v(\sigma)$ for all $v = \prod x_\alpha(r_\alpha)$ for which $\alpha({}^c l_1) r_{W\alpha} = r_{W\alpha}$ for all $\alpha \in \Phi^-_{U'} - \Phi^-_{Y'}$. The latter condition amounts to $\pi^{k_\alpha({}^c l_1)} \cdot r_{W\alpha} = 0$ for all $\alpha \in \Phi^-_{U'} - \Phi^-_{Y'}$.

But if $k_\alpha({}^c l_1) < h$ then $k'_\alpha = k_\alpha({}^c l_1)$, and if $k_\alpha({}^c l_1) = h$ then $k'_\alpha = h-1$, so since $\pi | r_\alpha$ already, the above condition amounts to $\pi^{k'_\alpha} \cdot r_{W\alpha} = 0$ for all $\alpha \in \Phi^-_{U'} - \Phi^-_{Y'}$, and so the lemma is proved in this case.

Case 2 :

Let $a = \prod x_\alpha(a_\alpha)$ with $\pi | a_\alpha$ for all α. Then writing $m = \prod x_\alpha(m_\alpha)$ we have $\pi^{h-1} | m_\alpha$ for all α, so by §1.34, $am = ma$.

Also $\pi_{h-1}(l) = 1$ implies $k_\alpha(l) \geq h-1$ for all α so $\alpha(l).a_\alpha = a_\alpha$ for all α, whence $lal^{-1} = \prod(l.x_\alpha(a_\alpha).l^{-1}) = \prod x_\alpha(\alpha(l).a_\alpha) = a$, i.e. $la = al$. (Incidentally, if $h \geq 2$, with $a = m$ we have $lm = ml$). Hence $ta = at$.

Now let $v = \prod x_\alpha(r_\alpha) \in U(w)$. Then $v.\theta_1(\sigma) = h.\theta_v(\sigma)$ where $h = \prod x_\alpha(h_\alpha)$ with $\pi | h_\alpha$, as may be seen by writing $\theta_1(\sigma)$ in distinguished form $uw.\rho_Y$, using the commutator formulae (§1.34) and noting that $\pi | r_\alpha$ for all α. Hence $t.\theta_1(\sigma) = \theta_1(\sigma)$ iff $vt.\theta_1(\sigma) = v.\theta_1(\sigma)$ iff $tv.\theta_1(\sigma) = v.\theta_1(\sigma)$ iff $th.\theta_v(\sigma) = h.\theta_v(\sigma)$ iff $ht.\theta_v(\sigma) = h.\theta_v(\sigma)$ iff $t\theta_v(\sigma) = \theta_v(\sigma)$; so t fixes $\theta_1(\sigma)$ iff t fixes $\theta_v(\sigma)$ for all $v \in U(w)$.

But $k'_\alpha = h-1$ for all α and $\pi | r_{w\alpha}$, so $r_{w\alpha}\pi^{k'_\alpha} = 0$ for all $\alpha \in \Phi^-_{U'} - \Phi^-_{Y'}$. Thus it remains only to show that :

$$t.\sigma = \sigma \quad \text{iff} \quad t.\theta_1(\sigma) = \theta_1(\sigma). \qquad (**)$$

Now $t.\theta_1(\sigma) = \theta_1(\sigma)$ iff $g^{-1}tg \in P_Y$ where $g = uw$, and $t.\sigma = \sigma$ iff $g^{-1}tg \in P_U$; so since $P_Y \subset P_U$ the implication $\Leftarrow$ of $(**)$ is clear. To prove $\Rightarrow$ we shall assume that $g^{-1}tg \in P_U$ and shall show then that in fact $g^{-1}tg \in P_Y$.

We have $l_1 = w^{-1}lw \in L_Y$ and $m_1 = w^{-1}mw \in {}^{w^{-1}}U_X$. Write $m_1 = m_2m_3$ (uniquely) with $m_2 \in U^- \cap {}^{w^{-1}}U_X = U^-_U \cap {}^{w^{-1}}U_X$, $m_3 \in U \cap {}^{w^{-1}}U_X$ and write (uniquely) $u_1^{-1} = v_1v_2$ with $v_2 \in U^-_U \cap {}^{w^{-1}}U$, $v_1 = \prod x_\alpha(r_\alpha) \in U^-_U \cap {}^{w^{-1}}U^-$, whence $\pi | r_\alpha$ and so $m_3v_1 = v_1m_3$.

But $m_3 \in U \cap {}^{w^{-1}}U$, $v_2 \in U^-_U \cap {}^{w^{-1}}U = U^- \cap {}^{w^{-1}}U$, so $m_3v_2 = v_2'm_3'$ with $v_2' \in U^- \cap {}^{w^{-1}}U$, $m_3' \in U \cap {}^{w^{-1}}U$. Also $u_1l_1m_2 = l_1'u_1'$ with $u_1' \in U^-_U$ by §1.27, since $l_1 \in L_Y \subseteq P'_U$ and $U^-_U \trianglelefteq P'_U$.

Hence $g^{-1}tg = u_1 l_1 m_1 u_1^{-1} = l_1.(u_1' v_1 v_2').m_3'$ with $l_1 \in P_Y$, $m_3' \in P_Y$, $u_1' v_1 v_2' \in U_U^-$, so if $g^{-1}tg \in P_U$ then $u_1' v_1 v_2' = l_1^{-1}.(g^{-1}tg).m_3'^{-1} \in P_U$, since $P_Y \subset P_U$, and so $u_1' v_1 v_2' \in P_U \cap U_U^- = 1$ and $g^{-1}tg = l_1 m_3'$ which belongs to P_Y. ∎

Proposition 2.27 is now proved, and so § 2.25/6 are also established.

Because of §2.24(ii), §2.26 completes the proof of theorem 2.22. To complete the proof of theorem 2.21 we must first improve §2.26 for t in Case 1 :

<u>Proposition 2.29</u>

$$S_G(t) = S_{L_X}(t).|\{\sigma \in \mathcal{F}_{w_0(X)}(w_1) \text{ fixed by } t\}|.$$

(This also holds for t in case 2 of course).

<u>Proof</u>:

We have ${}^{w_1}L_{w_0(X)} = L_X$ so $S_{L_{w_0(X)}(h-1)}(w_1^{-1}tw_1) = S_{L_X(h-1)}(t)$; and $|X'| = |w_0(X)'|$. So by § 2.26

$$S_G(t) = (-1)^{|X'|} S_{L_X(h-1)}(t).|\{\sigma \in \mathcal{F}_{w_0(X)}(w_1) \text{ fixed by } t\}|.$$

But $S_{L_X} = \sum_{X \subset Z} (-1)^{|Z'|} (S_{L_Z(h-1)})^{L_X}_{P_Z \cap L_X}$ since the standard parabolic subgroups of L_X are $P_Z \cap L_X$ $(X \subset Z)$ with standard Levi components L_Z (see § 2.1 and § 1.2). However the maximality of X implies that t can fix no parabolic of type Z for $X \subsetneq Z$, so can fix no $P_Z \cap L_X$ for $X \subsetneq Z$, so we have

$$S_{L_X} = (-1)^{|X'|} S_{L_X(h-1)}$$ and § 2.29 follows. ∎

<u>If X = ∅</u> then $L_X = G$ and $S_G(t) = (-1)^{|S|} S_{G(h-1)}(t)$. Also, since no proper parabolic subgroup of G contains t, $k_\alpha(t) = k_\alpha(\pi_{h-1}(t))$, (as $k_\alpha(t) < h$). By inductive hypothesis (which

was introduced in the proof of §2.25, Case 1) we have

$$S_{G(h-1)}(t) = (-1)^{\sum_{i=1}^{h-1} |X(i)'|} . q^{\sum_{\alpha \in \Phi^+} k_\alpha(\pi_{h-1}(t))} \text{, whence}$$

$$S_G(t) = (-1)^{\sum_{i=1}^{h} |X(i)'|} . q^{\sum_{\alpha \in \Phi^+} k_\alpha(t)}$$

, and §2.21 is proved in this case.

<u>If $X \neq \emptyset$</u> then we need the following further inductive hypothesis : §2.21 holds for groups of the form GL(W), W a free R-module of rank < n; then it holds for L_X (use §1.24), i.e. $S_{L_X}(t) = (-1)^x . q^{k(X)}$, writing $k(X) = \sum_{\alpha \in \Phi^+_{X'}} k_\alpha(t)$ (note the sign $(-1)^x$ is correct). The induction starts trivially with the case n = 2.

§2.29 and §2.24(i) now complete the proof. ∎

This completes the proof of theorems 2.1/2.

<u>Remarks</u>

The foregoing proof is technically much simplified if h = 1. In this case it also appears to constitute a new method of obtaining the known results ([17], §5.7) on the character of S_G.

Note that when h = 1 we have $U(w) = 1$ and $\theta_1(\mathcal{F}_U(w)) = \mathcal{F}_Y(w)$ (θ_1 is bijective). So we have a 'natural' injection from the set $\mathcal{F}_U$ of simplexes of type U in the Tits building T(V) of G to the set of simplexes $\mathcal{F}_Y$ of type Y, by means of which we could identify $\mathcal{F}_U$ as the subset $\bigcup_{w \in W_{U'}} \mathcal{F}_S(w)$ of $\mathcal{F}_S$. Curtis' formula now suggests that S_G should look like the permutation representation of G on $\sum_{U \subset S} (-1)^{|U'|} \bigcup_{w \in W_{U'}} \mathcal{F}_S(w) = \mathcal{F}_S(w_0)$ (§1.45), the set of simplexes of type S opposite to the standard one. (We give a precise result of this nature in §2.3). This may be compared with the ideas behind Tits' computation of the homology of T(V) ([17], §5.4).

2.3 The restriction of S_G to a parabolic subgroup

We retain the notation of § 2.2.

Theorem 2.31

Let t be either Case 1 or Case 2.

Then $\quad S_G(t) \;=\; (S_{L_X})^{P_X}_{L_X}(t).$

Proof:

Using the elements of U_X as a set of coset representatives of L_X in P_X (§1.27) we have

$$(S_{L_X})^{P_X}_{L_X}(t) = \sum_{u^{-1}tu \in L_X} S_{L_X}(u^{-1}tu).$$

In Case 1 $u^{-1}tu = t.[t,u] \in L_X$ iff $[t,u] = 1$ in which case $u^{-1}tu = t$ and so by §2.29 and §2.24(i)

$(S_{L_X})^{P_X}_{L_X}(t) = S_{L_X}(t).|\{u \in U_X : [t,u] = 1\}| = S_G(t).$

In Case 2, if $u^{-1}tu \in L_X$ then $l^{-1}u^{-1}lmu = l^{-1}u^{-1}tu \in L_X \cap U_X = 1$, so $m = l^{-1}.ulu^{-1} \in L_X.{}^{u}L_X \cap U_X = 1$, contrary to the hypothesis that $m \neq 1$. So $(S_{L_X})^{P_X}_{L_X}(t) = 0 = S_G(t)$ by § 2.22. ∎

In fact § 2.31 holds for any element t satisfying § 2.29. To see this, first note that for any $t \in P_X$, by §1.26/7,

$(S_{L_X})^{P_X}_{L_X}(t) = S_{L_X}(t).|\{\text{Levi subgroups of } P_X \text{ fixed by } t\}|.$

The connection with §2.29 is now established by :

Proposition 2.32

(i) $\mathfrak{F}_{w_0}(X)^{(w_1)} = \{\sigma \in \mathfrak{F}_{w_0}(X) \text{ opposite to } \rho_X\}$

(ii) There exists a P_X-equivariant bijection

$\{\sigma \in \mathfrak{F}_{w_0}(X) \text{ opposite to } \rho_X\} \to \{\text{Levi subgroups of } P_X\}.$

Proof:

(i) Let $\sigma = (0 \subset U_{i_1} \subset \ldots \subset U_{i_r} \subset V) \in \mathfrak{F}_{w_0}(X)$. Now w_1 acts on $\{1,2,..,n\}$ as follows : it reverses the order of the blocks $\{i : i_{j-1} < i \leq i_j\}$ $(1 \leq j \leq r+1)$, keeping the order within each block intact; hence $j \leq i_k$ implies $w_1(j) > n-i_k$. So if σ lies over w_1 we have $\overline{U}_{i_k} \cap \overline{V}_{n-i_k} = 0$ $(1 \leq k \leq r+1)$. Hence $U_{i_k} \cap V_{n-i_k} = 0$; for if not then since $U_{i_k} \sim (h^{i_k})$ and $V_{n-i_k} \sim (h^{n-i_k})$ we have $U_{i_k} + V_{n-i_k} \neq V$, whence $\overline{U}_{i_k} \oplus \overline{V}_{n-i_k} \neq \overline{V}$, contrary to the fact that both sides are vector spaces of dimension n. Thus σ is opposite to ρ_X.

Conversely, if σ is opposite to ρ_X then certainly $\overline{\sigma}$ lies in the $\overline{B}$-orbit on $\overline{\mathcal{F}_{w_0(X)}}$ given by the coset in W of $W_{w_0(X)}$, containing w_0; so σ lies over $w_1 \in w_0 W_{w_0(X)}$.
(ii) is just the composite of the bijections of §1.22/6. ∎

<u>Remarks</u>

(i) If t is split semisimple (so $X = S$ and $S_{I_X} = 1$) $S_G(t) = |\{\sigma \in \mathcal{F}$ opposite ρ_S and fixed by $t\}|$, so on such elements S_G behaves like the permutation representation of G on the set of minimal parabolic subgroups 'opposite' to a fixed one.
(ii) If $h = 1$ then since §[illegible].21/2 compute the character of S_G completely we have $S_G|_P = (S_L)_L^P$, where P is any parabolic subgroup of G and L a Levi component of P.

However, in case $h \geq 2$ this appears to be false, even if $n = 2$; for example, if $h = 2$ then $S_G\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = q$, whereas $(S_T)_T^B\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = 0$. For more details see Chapter 5.

Further properties of S_G will be discussed in Chapters 4 and 5.

Chapter 3

The irreducible Steinberg representations

In this chapter we construct, for $h \geq 2$, a family of irreducible subrepresentations $R(a)$ of 1_B^G each of which is realisable as the homology of a suitable simplicial complex (§3.2). One of them, denoted St_G, is the 'largest' component of 1_B^G and may be regarded as an analogue of the usual Steinberg representation S_G of G in the case $h=1$. (We defer more detailed connection between the two cases until Chapter 4). However, the character of St_G vanishes on regular unramified semisimple elements, quite unlike that of S_G in case $h=1$.

We also consider, for $h \geq 2$, an irreducible 'affine Steinberg representation' St_H of the affine group H (§3.3/4); it may be constructed homologically, in a like manner to St_G, or alternatively as the representation χ_U^H of H induced from a non-singular character χ of U, and this leads to the fact that St_G is contained in χ_U^G with multiplicity 1. In §3.5 we show that χ_U^G is multiplicity-free if $n=2$ or 3 (at least), but if $n=3$ and $h \geq 2$ then the number of components of χ_U^G exceeds the number of regular conjugacy classes of G, unlike the case $n=2$, or the case $h=1$, where these numbers are equal.

We also show how St_G fits into the 'principal series' of G (§3.6).

3.1 Preliminaries concerning flags

Let W be an R-module of type $(a_1,\dots,a_m)$. Then $|W| = q^{|\underline{a}|}$ where $|\underline{a}| = \sum_{i=1}^{m} a_i$ (§1.04). We are interested in maximal proper submodules of W : these have index q in W.

Define $K = \pi W = \{\pi x : x \in W\}$; then $\overline{W} = W/K$ is a vector space over $\overline{R}$ of dimension $m = l(W)$. Also let $U = \{x \in W : \pi^{h-1}x = 0\}$; note that $K \subset U$.

Lemma 3.11

(i) The quotient map $\pi : W \to \overline{W}$ induces a canonical bijection

$$\{\text{index } q \text{ submodules of } W\} \to \{\text{hyperplanes in } \overline{W}\}.$$

(ii) Assume that $h \geq 2$ and suppose $W \sim (h^{m-1}, a)$ with $m \geq 2$ and $1 \leq a \leq h-1$. Then $L = U/K$ is a canonically defined line in $\overline{W}$ such that in the bijection of (i) we have

$$\{M \subset W : M \sim (h^{m-1}, a-1)\} \to \{\text{hyperplanes in } \overline{W} \text{ not containing } L\}$$

$$\{M \subset W : M \sim (h^{m-2}, h-1, a)\} \to \{\text{hyperplanes in } \overline{W} \text{ containing } L\}.$$

Proof:

If M is a submodule of W of index q then $M \sim (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_m)$ for some i, and $\pi x \in M$ for all $x \in W$, i.e. $K \subset M$. Moreover π induces a canonical bijection

$$\{\text{submodules of } W \text{ containing } K\} \to \{\text{subspaces of } \overline{W}\}$$

and $|\overline{W}|/|\pi(M)| = |W|/|M| = q$, whence $\pi(M)$ is a hyperplane in $\overline{W}$.

In case $W \sim (h^{m-1}, a)$ with $m \geq 2$ and $a > 0$, M has type either $(h^{m-1}, a-1)$ or $(h^{m-2}, h-1, a)$; these are different if $h \geq 2$ and $a < h$, and we have $M \sim (h^{m-1}, a-1)$ iff there exists $x \in W$, of order π^a, which is not in M iff $U \not\subset M$ iff $L \not\subset \pi(M)$.

Finally if $x \in U$ but $x \notin K$ then x has order π^b with $b < h$; and $\pi x \in K$. So since W can be generated by elements $e_1, \ldots, e_{m-1}$ of order π^h and e_m of order π^a we see $a < h$ implies that $|U|/|K| = q$, whence L is a line in $\overline{W}$. ∎

Corollary 3.12

(i) Let $l(W) = m$. Then W has $(q^m - 1)/(q-1)$ index q submodules.

(ii) Assume $h \geq 2$ and suppose $W \sim (h^{m-1}, a)$ with $m \geq 2$ and $1 \leq a \leq h-1$. Then W contains q^{m-1} submodules of type $(h^{m-1}, a-1)$.

Proof:

(i) $m = l(W) = \dim_{\bar{R}}(\bar{W})$ and so the number of hyperplanes in $\bar{W}$ is $(q^m-1)/(q-1)$; now use §3.11(i).

(ii) $|\{\text{hyperplanes in } \bar{W} \text{ containing a given line}\}|$

$= |\{\text{lines in } \bar{W} \text{ contained in a given hyperplane}\}|$, by the usual duality,

$= (q^{m-1}-1)/(q-1)$.

So $|\{\text{submodules of } W \text{ of type } (h^{m-1},a-1)\}|$

$= (q^m-1)/(q-1) - (q^{m-1}-1)/(q-1) = q^{m-1}$. ∎

Remark

Dually, one may consider quotients $\theta: W \to U$ with $|\ker\theta| = q$ and count them in the same way. Explicitly, define $\tilde{W} = \{x \in W : \pi x = 0\}$ (a vector space over $\pi^{h-1}R$). Then there exists a canonical bijection

$\{\text{'index } q \text{ quotients' of } W\} \to \{\text{lines in } \tilde{W}\}$,

and if $h \geq 2$ and $W \sim (h^{m-1},a)$ with $m \geq 2$, $1 \leq a \leq h-1$, then $H = \pi^{h-1}W$ is a canonically defined hyperplane in $\tilde{W}$ such that

$\{\text{quotients of } W \text{ of type } (h^{m-1},a-1)\} \to \{\text{lines } \ell \text{ in } \tilde{W} : \ell \not\subset H\}$

$\{\text{quotients of } W \text{ of type } (h^{m-2},h-1,a)\} \to \{\text{lines } \ell \text{ in } \tilde{W} : \ell \subset H\}$.

We shall assume that $h \geq 2$ for the remainder of this section.

Lemma 3.13

Let $A \sim (h^r,a)$ with $0 \leq a \leq h$ and $B \sim (h^{r+1},b)$ with $1 \leq b \leq h$ and $A \subset B$; assume that if $b=h$ then $a > 0$.

Suppose there exists $C \sim (h^{r+1})$ with $A \subset C \subset B$. Then

$$|\{D \sim (h^{r+1},b-1) : \text{there exists } E \sim (h^{r+1}) \text{ with } A \subset E \subset D \subset B\}| = \begin{cases} q & \text{if } a+b \leq h \\ 1 & \text{if } a+b > h \end{cases}.$$

Proof:

By choosing $F \subset A$ with $F \sim (h^r)$ and factoring it out, we may reduce the problem to the case $r=0$.

The existence of C implies that there exists $x,y \in B$ with x of order π^h and y of order π^b such that $B = \langle x,y \rangle$ and $A = \langle \pi^{h-a}x \rangle$.

First assume that $b < h$. Then by §3.12(ii) B has q submodules D of type $(h,b-1)$ and these are $D_r = \langle x+\tilde{r}y, \pi y \rangle$ $(r \in \bar{R})$. Then $D_r \subset A$ iff $\pi^{h-a}\tilde{r}y = 0$ iff $\tilde{r} = 0$ or $h-a \geq b$. Hence

$$|\{D \sim (h,b-1) : A \subset D \subset B\}| = \begin{cases} q & \text{if } a+b \leq h \\ 1 & \text{if } a+b > h \end{cases}.$$

But D_r always contains $E = \langle x+\tilde{r}y \rangle \sim (h)$ and this contains A if $r = 0$ or $h-a \geq b$, so the lemma is proved if $b < h$.

Now assume $b = h$. Then by §3.12(i) there are q+1 modules $D \sim (h,h-1)$ with $A \subset D \subset B$, viz. $D_r = \langle x+\tilde{r}y, \pi y \rangle$ $(r \in \bar{R})$ and $\langle y, \pi x \rangle$. In case $E \sim (h)$ with $E \subset D = \langle y, \pi x \rangle$, E must have the form $\langle y+s.\pi x \rangle$ for some $s \in R$; but then $A \not\subset E$ if $a > 0$, so we may ignore $D = \langle y, \pi x \rangle$ and proceed with the computation as before. ∎

There is a 'dual' of §3.13 as follows.

<u>Lemma 3.14</u>

Let $A \sim (h^r,a)$ with $0 \leq a \leq h-1$ and $B \sim (h^{r+1},b)$ with $0 \leq b \leq h$ and $A \subset B$; assume that if $a = 0$ then $b < h$.

Suppose there exists $C \sim (h^{r+1})$ with $A \subset C \subset B$. Then

$$|\{D \sim (h^r,a+1) : \text{there exists } E \sim (h^{r+1}) \text{ with } A \subset D \subset E \subset B\}| = \begin{cases} q & \text{if } a+b \geq h \\ 1 & \text{if } a+b < h \end{cases}.$$

<u>**Proof:**</u>

Reduce to the case $r = 0$ as in §3.13, and consider B as a submodule of a module $M \sim (h,h)$. Then we have injective maps $M/B \to M/C \to M/A$ given by $m+B \to m+C \to m+A$ $(m \in M)$ which may be considered as inclusions; we require modules $D' = M/D \sim (h,h-a-1)$ such that there exists $E' = M/E \sim (h)$ with

$M/B = B' \subset E' \subset D' \subset A' = M/A$. By §3.13 the number of these is $\begin{cases} q & \text{if } (h-b)+(h-a) \leq h \\ 1 & \text{if } (h-b)+(h-a) > h \end{cases} = \begin{cases} q & \text{if } a+b \geq h \\ 1 & \text{if } a+b < h \end{cases}$. ∎

As before, V denotes a free R-module of rank $n \geq 2$. Consider flags $\mathcal{Y} = (0 \subset U_1 \subset .. \subset U_{n-1} \subset V)$ of submodules of V subject to the conditions

(i) $U_i \sim (h^{i-1}, a_i)$ for some a_i such that $0 \leqslant a_i \leqslant h$ $(1 \leqslant i \leqslant n-1)$

(ii) there exists $V_i \sim (h^i)$ with $U_i \subset V_i \subset U_{i+1}$ $(1 \leqslant i \leqslant n-1)$

(iii) $a_i + a_{i+1} \geq h$ $(0 \leqslant i \leqslant n-1)$

(Conventionally we take $U_0 = 0$, $U_n = V$, and $a_o = h = a_n$).

Such a flag will be said to have type $(a_1,..,a_{n-1}) = \underset{\sim}{a}$, and $\mathcal{F}(\underset{\sim}{a})$ will denote the set of all flags of type $\underset{\sim}{a}$. We write $|\underset{\sim}{a}| = \sum_{i=1}^{n-1} a_i$. Also $(h,h,..,h) = (h^{n-1})$; note $\mathcal{F}(h^{n-1}) = \mathcal{F}_S$.

The following example shows that condition (ii) is not redundant : let $n=2$, $V = \langle x,y \rangle$ and assume $h \geq 2$. Let $U_1 = \langle \pi x \rangle$ and $U_2 = \langle x+y, \pi y \rangle$; so $U_1 \sim (h-1)$ and $U_2 \sim (h,h-1)$. Then any submodule W of U_2 of type (h) has the form $\langle x+y+r.\pi y \rangle$ for some $r \in R$, and contains U_1 provided $\pi x \in \langle x+y+r.\pi y \rangle$, i.e. provided $\pi y + r.\pi^2 y = 0$: this is impossible since y has order π^h and $h \geq 2$. So there does not exist $V_1 \sim (h)$ with $U_1 \subset V_1 \subset U_2$.

We partially order these flags as follows.

Let $\mathcal{Y} = (0 \subset U_1 \subset .. \subset U_{n-1} \subset V)$, $\mathcal{Y}' = (0 \subset U_1' \subset .. \subset U_{n-1}' \subset V)$.

Then $\mathcal{Y} \geq \mathcal{Y}'$ iff $U_i' \subset U_i$ and there exists $W_{i-1} \sim (h^{i-1})$ such that $U_{i-1} \subset W_{i-1} \subset U_i'$ $(1 \leqslant i \leqslant n-1)$ $(U_0 = W_0 = 0)$.

Note that if $\mathcal{Y} \sim \underset{\sim}{a} = (a_1,..,a_{n-1})$, $\mathcal{Y}' \sim \underset{\sim}{a}' = (a_1',..,a_{n-1}')$ and $\mathcal{Y} \geq \mathcal{Y}'$ then $\underset{\sim}{a} \geq \underset{\sim}{a}'$, i.e. $a_i \geq a_i'$ $(1 \leqslant i \leqslant n-1)$.

From now on $\underset{\sim}{a} = (a_1,\ldots,a_{n-1})$ and it will be convenient for us to write $(\underset{\sim}{a},J) = (b_1,\ldots,b_{n-1})$ where $b_i = \begin{cases} a_i & \text{if } i \in J \\ a_i - 1 & \text{if } i \notin J \end{cases}$ for each $J \subset S$.

Proposition 3.15

(i) Suppose given $i \in S$ with $a_{i-1}+a_i > h$, and $\mathcal{F} \in \mathcal{F}(\underset{\sim}{a})$. Then there exists a unique $\mathcal{F}_i \in \mathcal{F}(\underset{\sim}{a},\{i\}')$ such that $\mathcal{F} \geq \mathcal{F}_i$.

(ii) Suppose given $i \in S$ with $a_i \geq 2$ and $a_i+a_{i+1} > h$, and $\mathcal{F}_i \in \mathcal{F}(\underset{\sim}{a},\{i\}')$. Then there exist precisely q flags $\mathcal{F} \in \mathcal{F}(\underset{\sim}{a})$ such that $\mathcal{F} \geq \mathcal{F}_i$.

Proof:

Let $\mathcal{F} = (0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$ and put $A = U_{i-1} \sim (h^{i-2}, a_{i-1})$ (if i=1 then A=0) and $B = U_i \sim (h^{i-1}, a_i)$. We know there exists $C = V_{i-1} \sim (h^{i-1})$ such that $A \subset C \subset B$. Putting $a=a_{i-1}$, $b=a_i$ we have $a+b > h$ (in particular $b \geq 1$ and if $b=h$ then $a > 0$). So §3.13 implies there exists a unique $D \sim (h^{i-1}, a_i - 1)$ such that there exists $E \sim (h^{i-1})$ with $A \subset E \subset D \subset B$. Hence $\mathcal{F}_i = (0 \subset U_1 \subset \ldots \subset U_{i-1} \subset D \subset U_{i+1} \subset \ldots \subset U_{n-1} \subset V)$ is the unique flag of type $(\underset{\sim}{a},\{i\}')$ which is $\leq \mathcal{F}$.

(ii) Let $\mathcal{F}_i = (0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$ and put $A = U_i \sim (h^{i-1}, a_i - 1)$ and $B = U_{i+1} \sim (h^i, a_{i+1})$ (if i=n-1 then B=V). We know there exists $C = V_i \sim (h^i)$ such that $A \subset C \subset B$. Putting $a=a_i-1$, $b=a_{i+1}$ we have $a+b \geq h$ and $1 \leq a \leq h-1$. So §3.14 implies there exist precisely q modules $D \sim (h^{i-1}, a_i)$ such that there exists $E \sim (h^i)$ with $A \subset D \subset E \subset B$. Hence there exist precisely q flags $\mathcal{F} = (0 \subset U_1 \subset \ldots \subset U_{i-1} \subset D \subset U_{i+1} \subset \ldots \subset U_{n-1} \subset V) \in \mathcal{F}(\underset{\sim}{a})$ such that $\mathcal{F} \geq \mathcal{F}_i$. ■

In practice the result we shall really need is the following corollary of §3.15.

Corollary 3.16

Let $1 \leq a_i \leq h$ and $a_i + a_{i+1} \geq h$ $(1 \leq i \leq n-1)$ and let $\underset{\sim}{b} \geq \underset{\sim}{a}$.

(i) For each $\mathcal{J} \in \mathcal{F}(\underset{\sim}{b})$ there exists a unique $\mathcal{G} \in \mathcal{F}(\underset{\sim}{a})$ with $\mathcal{J} \geq \mathcal{G}$; and for each $\mathcal{G} \in \mathcal{F}(\underset{\sim}{a})$ there exist precisely $q^{|\underset{\sim}{b}|-|\underset{\sim}{a}|}$ flags $\mathcal{H} \in \mathcal{F}(\underset{\sim}{b})$ with $\mathcal{H} \geq \mathcal{G}$.

(ii)
$$|\mathcal{F}(\underset{\sim}{a})| = q^{n-1} . \prod_{i=1}^{n-1} \left(q^{a_i - h} . \frac{q^{ih} - q^{ih-(i+1)}}{q-1}\right)$$

$$= \prod_{i=1}^{n-1} \frac{q^{i+1}-1}{q-1} . q^{i(h-1)+a_i-h} \quad .$$

Proof:

(i) is immediate from §3.15

For (ii) it is sufficient to prove the special case $\underset{\sim}{a} = (h^{n-1})$ and then use the fact $|\mathcal{F}(\underset{\sim}{a})| = q^{|\underset{\sim}{a}|-|(h^{n-1})|} . |\mathcal{F}(h^{n-1})|$, which follows from (i).

Now given $A \sim (h^i)$, $B \sim (h^{i+1})$ with $A \subset B$, §3.13 shows there exists a unique $W_j \sim (h^i, j)$ with $A \subset W_j \subset B$ for each j, $0 \leq j \leq h$, and moreover $W_{j-1} \subset W_j$ $(1 \leq j \leq h)$. Hence

$$\left|\{A \sim (h^i) \text{ contained in a given } B \sim (h^{i+1})\}\right|$$
$$= \prod_{j=0}^{h-1} \left|\{W_j \sim (h^i, j) \text{ contained in a given } W_{j+1} \sim (h^i, j+1)\}\right|$$
$$= \frac{q^{i+1}-1}{q-1} . \prod_{j=0}^{h-2} q^i \quad = \frac{q^{i+1}-1}{q-1} . q^{i(h-1)} .$$

$$\text{So } |\mathcal{F}(h^{n-1})| = \prod_{i=1}^{n-1} \left|\{U_i \sim (h^i) \text{ contained in a given } U_{i+1} \sim (h^{i+1})\}\right|$$
$$= \prod_{i=1}^{n-1} q^{i(h-1)} . \frac{q^{i+1}-1}{q-1}$$
$$= q^{n-1} . \prod_{i=1}^{n-1} \frac{q^{ih} - q^{ih-(i+1)}}{q-1} \quad . \quad \blacksquare$$

Fix $\mathcal{J}_{\underset{\sim}{a}} \in \mathcal{F}(\underset{\sim}{a})$. G acts on $\mathcal{F}(\underset{\sim}{a})$ by left translation; write $B_{\underset{\sim}{a}} = \mathrm{Stab}_G(\mathcal{J}_{\underset{\sim}{a}})$. An almost identical argument to that of §1.11 yields :

Corollary 3.16

Let $1 \leq a_i \leq h$ and $a_i + a_{i+1} \geq h$ $(1 \leq i \leq n-1)$ and let $\underset{\sim}{b} \geq \underset{\sim}{a}$.

(i) For each $\mathfrak{J} \in \mathfrak{F}(\underset{\sim}{b})$ there exists a unique $\mathfrak{G} \in \mathfrak{F}(\underset{\sim}{a})$ with $\mathfrak{J} \geq \mathfrak{G}$; and for each $\mathfrak{G} \in \mathfrak{F}(\underset{\sim}{a})$ there exist precisely $q^{|\underset{\sim}{b}|-|\underset{\sim}{a}|}$ flags $\mathfrak{H} \in \mathfrak{F}(\underset{\sim}{b})$ with $\mathfrak{H} \geq \mathfrak{G}$.

(ii) $$|\mathfrak{F}(\underset{\sim}{a})| = q^{n-1} . \prod_{i=1}^{n-1} \left(q^{a_i - h} . \frac{q^{ih} - q^{ih-(i+1)}}{q-1}\right)$$

$$= \prod_{i=1}^{n-1} \frac{q^{i+1}-1}{q-1} . q^{i(h-1)+a_i-h} .$$

Proof:

(i) is immediate from §3.15

For (ii) it is sufficient to prove the special case $\underset{\sim}{a} = (h^{n-1})$ and then use the fact $|\mathfrak{F}(\underset{\sim}{a})| = q^{|\underset{\sim}{a}|-|(h^{n-1})|} . |\mathfrak{F}(h^{n-1})|$, which follows from (i).

Now given $A \sim (h^i)$, $B \sim (h^{i+1})$ with $A \subset B$, §3.13 shows there exists a unique $W_j \sim (h^i, j)$ with $A \subset W_j \subset B$ for each j, $0 \leq j \leq h$, and moreover $W_{j-1} \subset W_j$ $(1 \leq j \leq h)$. Hence

$$|\{A \sim (h^i) \text{ contained in a given } B \sim (h^{i+1})\}|$$
$$= \prod_{j=0}^{h-1} |\{W_j \sim (h^i, j) \text{ contained in a given } W_{j+1} \sim (h^i, j+1)\}|$$
$$= \frac{q^{i+1}-1}{q-1} . \prod_{j=0}^{h-2} q^i = \frac{q^{i+1}-1}{q-1} . q^{i(h-1)} .$$

So $$|\mathfrak{F}(h^{n-1})| = \prod_{i=1}^{n-1} |\{U_i \sim (h^i) \text{ contained in a given } U_{i+1} \sim (h^{i+1})\}|$$
$$= \prod_{i=1}^{n-1} q^{i(h-1)} . \frac{q^{i+1}-1}{q-1}$$
$$= q^{n-1} . \prod_{i=1}^{n-1} \frac{q^{ih} - q^{ih-(i+1)}}{q-1} . \quad \blacksquare$$

Fix $\mathfrak{J}_{\underset{\sim}{a}} \in \mathfrak{F}(\underset{\sim}{a})$. G acts on $\mathfrak{F}(\underset{\sim}{a})$ by left translation; write $B_{\underset{\sim}{a}} = \mathrm{Stab}_G(\mathfrak{J}_{\underset{\sim}{a}})$. An almost identical argument to that of §1.11 yields :

Lemma 3.17

There exists a G-equivariant bijection $G/B_{\underset{\sim}{a}} \to \mathcal{F}(\underset{\sim}{a})$ given by $gB_{\underset{\sim}{a}} \to g.\mathcal{F}_{\underset{\sim}{a}}$ $(g \in G)$. ■

We shall suppose the basis $\underset{\sim}{e} = \{e_1,\ldots,e_n\}$ of V is such that for each $\underset{\sim}{a}$, $\mathcal{F}_{\underset{\sim}{a}} = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$ with $V_i = \langle e_1,\ldots,e_{i-1},\pi^{h-a_i}e_i \rangle$ (this is possible by condition (ii) in the definition of flags of type $\underset{\sim}{a}$). Assume $a_i \geq 1$ $(1 \leq i \leq n-1)$.

Let $\mathcal{B}_{\underset{\sim}{a}} = \{uw : w \in W,\ u = \prod_{\alpha<0} {}^{w}x_{\alpha}(r_{\alpha})$ with conditions :

if $w\alpha > 0$ then $r_\alpha \in R$; $r_\alpha \in R_{a_i}$ if also $-\alpha = \alpha_{i,i+1} \in \Sigma$,

if $w\alpha < 0$ then $r_\alpha \in \pi R$; $r_\alpha \in \pi R_{a_i}$ if also $-\alpha = \alpha_{i,i+1} \in \Sigma$. $\}$

Here R_k is identified with the subgroup $\{\sum_{0 \leq j \leq k-1} \tilde{r}_j \pi^j : r_j \in \bar{R}\}$ of R, and πR_k means $\{\sum_{1 \leq j \leq k-1} \tilde{r}_j \pi^j : r_j \in \bar{R}\}$.

Proposition 3.18

$|\mathcal{B}_{\underset{\sim}{a}}| = |G/B_{\underset{\sim}{a}}|$ and $\mathcal{B}_{\underset{\sim}{a}}$ forms a complete set of representatives for left cosets of $B_{\underset{\sim}{a}}$ in G.

Proof:

The case when $\underset{\sim}{a} = (h^{n-1})$ has already been proved : $B_{\underset{\sim}{a}} = B = P_S$ and §1.46 contains the result.

Now as a matrix group with respect to the basis $\underset{\sim}{e}$,

$$B_{\underset{\sim}{a}} = \{(a_{ij}) : a_{ij} \in R\ (1 \leq i < j \leq n),\ a_{ii} \in R^*\ (1 \leq i \leq n),$$
$$a_{i+1,i} \in \pi^{a_i}R\ (1 \leq i \leq n-1),\ a_{ij} = 0\ (2 \leq j+1 < i \leq n)\ \},$$
$$= \prod_{i=1}^{n-1} U_{i+1,i}(\pi^{a_i}R).B$$

So $|B_{\underset{\sim}{a}}|/|B| = |\prod_{i=1}^{n-1} U_{i+1,i}(\pi^{a_i}R)| = \prod_{i=1}^{n-1} q^{h-a_i}$.

Now $a_i \geq 1$ so $|R|/|R_{a_i}| = |\pi R|/|\pi R_{a_i}| = |\pi^{a_i}R| = q^{h-a_i}$ $(1 \leq i \leq n-1)$.

Hence $|\mathcal{B}_{(h^{n-1})}|/|\mathcal{B}_{\underset{\sim}{a}}| = |B_{\underset{\sim}{a}}|/|B|$, so using the result for the case when $\underset{\sim}{a} = (h^{n-1})$ we see $|\mathcal{B}_{\underset{\sim}{a}}| = |G|/|B_{\underset{\sim}{a}}| = |G/B_{\underset{\sim}{a}}|$.

We must now show that if $g_1, g_2 \in \mathcal{B}_{\underset{\sim}{a}}$ and $g_1 \neq g_2$ then $g_1B_{\underset{\sim}{a}} \neq g_2B_{\underset{\sim}{a}}$.

So suppose $g_1 = u_1w_1$, $g_2 = u_2w_2$. Now $\overline{g}_1 \in \overline{U}w_1$, so lies in the $\overline{B}$-$\overline{B}$ double coset of $\overline{G}$ given by w_1 (§1.47); and w_2 lies in the $\overline{B}$-$\overline{B}$ double coset given by w_2. So if $w_1 \neq w_2$ then $\overline{g}_1\overline{B} \neq \overline{g}_2\overline{B}$ and so $\overline{g}_1\overline{B}_{\underset{\sim}{a}} \neq \overline{g}_2\overline{B}_{\underset{\sim}{a}}$ (since $\overline{B} = \overline{B}_{\underset{\sim}{a}}$) and hence $g_1B_{\underset{\sim}{a}} \neq g_2B_{\underset{\sim}{a}}$.

Now assume $w_1 = w_2$, and suppose $g_1B_{\underset{\sim}{a}} = g_2B_{\underset{\sim}{a}}$, i.e. $g_2^{-1}g_1 \in B_{\underset{\sim}{a}}$. Then

$$
\begin{aligned}
g_2^{-1}g_1 &= w_1^{-1}u_2^{-1}u_1w_1 \\
&= w_1^{-1}\Big(\prod_{\alpha<0} w_1x_\alpha(r_\alpha)w_1^{-1}\Big)^{-1}.\Big(\prod_{\alpha<0} w_1x_\alpha(s_\alpha)w_1^{-1}\Big), \text{ say} \\
&= \prod_{\substack{\alpha<0 \\ -\alpha\notin Z}} x_\alpha(t_\alpha).\prod_{i=1}^{n-1} x_{i+1,i}(-r_i+s_i), \text{ by §1.34/5;}
\end{aligned}
$$

(here $r_i = r_{\alpha_{i+1,i}}$, $s_i = s_{\alpha_{i+1,i}}$ $(1\leq i\leq n-1)$, and $t_\alpha \in R$).

Thus $g_2^{-1}g_1 \in U^-$, so by §1.32 all $t_\alpha = 0$ and for $1\leq i\leq n-1$ we have $\pi^{a_i} \mid (-r_i+s_i)$. But r_i, s_i are both in R_{a_i} so $(-r_i+s_i)$ is in R_{a_i} so is zero. Hence $r_i = s_i$ $(1\leq i\leq n-1)$, and $g_1 = g_2$. ∎

Remarks

We could not phrase the above proof exactly as for §1.46 since $\overline{\mathfrak{F}_{\underset{\sim}{a}}} \neq \overline{\rho_S}$. But of course $\overline{B}_{\underset{\sim}{a}} = \overline{B}$.

Now there is a 'dual' formulation of this theory in which for example we replace V_i by $V_i' = \langle e_1,\ldots,e_i,\pi^{a_i}e_{i+1}\rangle$ of type $(h^i, h-a_i)$ $(1\leq i\leq n-1)$ and we replace $\mathfrak{F}_{\underset{\sim}{a}}$ by $\mathfrak{F}_{\underset{\sim}{a}}' = (0 \subset V_1' \subset \ldots \subset V_{n-1}' \subset V)$. Then $\mathrm{Stab}_G(\mathfrak{F}_{\underset{\sim}{a}}') = B_{\underset{\sim}{a}} = \mathrm{Stab}_G(\mathfrak{F}_{\underset{\sim}{a}})$, and $\overline{\mathfrak{F}_{\underset{\sim}{a}}'} = \overline{\rho_S}$.

More generally we may define the flag $(0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$ to be of type 'dual' to $\underset{\sim}{a}$ iff :

(i) $U_i \sim (h^i, h-a_i)$ $(1\leq i\leq n-1)$

(ii) there exists $W_i \sim (h^i)$ such that $U_{i-1} \subset W_i \subset U_i$ $(1\leq i\leq n-1)$.

Then there is a G-equivariant bijection

$$\{\text{flags of type } \underset{\sim}{a}\} \rightarrow \{\text{~~flags of~~ type 'dual' to } \underset{\sim}{a}\}$$

given by $\quad g.\mathfrak{F}_{\underset{\sim}{a}} \mapsto g.\mathfrak{F}_{\underset{\sim}{a}}' \quad (g \in G)$.

3.2 The representations $R(\underset{\sim}{a})$

Definition

The sequence of integers $\underset{\sim}{a} = (a_1,..,a_{n-1})$ will be called admissible if $2 \leq a_i \leq h$ $(1\leq i\leq n-1)$ and $a_i + a_{i+1} > h+1 (0\leq i\leq h-1)$.

Conventionally we put $a_0 = h = a_n$, as before. Notice that no sequence can be admissible if $h = 1$.

For each admissible sequence $\underset{\sim}{a}$ we construct a simplicial complex $X(\underset{\sim}{a})$ as follows.

(i) For $0\leq r\leq n-2$ the r-simplices are flags of type $(\underset{\sim}{a},J)$ where $|J| = r+1$.

(ii) For $0\leq r\leq s\leq n-2$ the r-simplex $\mathfrak{F}$ is a face of the s-simplex $\mathfrak{G}$ iff $\mathfrak{F} \leq \mathfrak{G}$ (in the sense of §3.1).

To see that this makes sense we first note that $X(\underset{\sim}{a})$ is the disjoint union of the subcomplexes $X_{\mathfrak{G}}(\underset{\sim}{a})$ consisting of all flags which are $\geq$ some fixed flag $\mathfrak{G}$ of type $(\underset{\sim}{a},\emptyset)$, since by §3.16(i) each flag in question is $\geq$ a unique $\mathfrak{G}$ of this type ($\underset{\sim}{a}$ admissible implies $(a_i-1)+(a_{i+1}-1) \geq h$).

Now the points of $X_{\mathfrak{G}}(\underset{\sim}{a})$ are those flags $\mathfrak{F}$ of type $(\underset{\sim}{a},\{i\})$ for some i, with $\mathfrak{F} \geq \mathfrak{G}$, and there are precisely q of these for each $i \in S$. Moreover if $0\leq r\leq n-2$ then an r-simplex $\mathfrak{F} \sim (\underset{\sim}{a},J)$ as defined above has precisely one vertex of each type $(\underset{\sim}{a},\{i\})$ with $i \in J$, and no others (so it really is an r-simplex) and any subset of the vertices of $\mathfrak{F}$, say those of types $(\underset{\sim}{a},\{i\})$ for $i \in K \subset J$, determines a unique face $\mathfrak{H} \leq \mathfrak{F}$, of type $(\underset{\sim}{a},K)$.

Also there exist just q^{r+1} r-simplices of a given type in $X_{\mathfrak{G}}(\underset{\sim}{a})$, so we see that any set of r+1 vertices in $X_{\mathfrak{G}}(\underset{\sim}{a})$ of distinct types must form an r-simplex (since there are just q vertices of each type). Hence $X_{\mathfrak{G}}(\underset{\sim}{a})$ is the join of n-1 sets, each containing the q vertices of a given type.

So we have proved:

Proposition 3.21

$X(\underset{\sim}{a})$ is the disjoint union $\bigcup_{\varrho \in \mathcal{F}(\underset{\sim}{a},\emptyset)} X_\varrho(\underset{\sim}{a})$ of $|\mathcal{F}(\underset{\sim}{a},\emptyset)|$ copies $X_\varrho(\underset{\sim}{a})$ of the join of n-1 copies of a discrete set of q points. ∎

Lemma 3.22

Let Y be the join of n-1 copies of a discrete set of q points. Then
$$\widetilde{H}_i(Y,\mathbb{Z}) \cong \begin{cases} 0 & \text{if } i \neq n-2 \\ \mathbb{Z}^{(q-1)(n-1)} & \text{if } i = n-2 \end{cases}$$

Proof:

For torsion-free spaces A,B (torsion-free homology) if $A * B$ denotes their join then
$$\widetilde{H}_i(A * B, \mathbb{Z}) \cong \bigoplus_{l+m=i-1} \widetilde{H}_l(A,\mathbb{Z}) \otimes_{\mathbb{Z}} \widetilde{H}_m(B,\mathbb{Z}) \quad ([14]).$$
Taking A to be a discrete set of q points we have
$$\widetilde{H}_i(A,\mathbb{Z}) \cong \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{Z}^{q-1} & \text{if } i = 0. \end{cases}$$
So if $Y = A * A * .. * A$ (n-1 copies) then
$$\widetilde{H}_i(Y,\mathbb{Z}) \cong \begin{cases} 0 & \text{if } i \neq n-2 \\ \underbrace{\mathbb{Z}^{q-1} \otimes \mathbb{Z}^{q-1} \otimes .. \otimes \mathbb{Z}^{q-1}}_{(\text{ n-1 copies })} & \text{if } i = n-2. \end{cases}$$ ∎

It follows from §3.22 that the augmented chain complex of Y is exact :
$$0 \to \widetilde{H}_{n-2}(Y,\mathbb{Z}) \to C_{n-2}(Y) \xrightarrow[d_{n-2}]{} \dots \xrightarrow[d_1]{} C_0(Y) \xrightarrow[\varepsilon]{} \mathbb{Z} \to 0 .$$

Here $C_i(Y)$ denotes the $\mathbb{Z}$-module of i-chains of Y, d_i are the usual boundary maps and ε the augmentation.

Now we take $Y = X_\varrho(\underset{\sim}{a})$ and take the direct sum of these exact sequences over all $\varrho \in \mathcal{F}(\underset{\sim}{a},\emptyset)$, obtaining : (‡)
$$0 \to R(\underset{\sim}{a}) \to C_{n-2}(X(\underset{\sim}{a})) \xrightarrow[d_{n-2}]{} \dots \xrightarrow[d_1]{} C_0(X(\underset{\sim}{a})) \xrightarrow[d_{-1}]{} C_{-1}(X(\underset{\sim}{a})) \to 0.$$

Here $C_i(X(\underset{\sim}{a}))$ $(i \geq 0)$ denotes the $\mathbb{Z}$-module of (oriented) i-chains of $X(\underset{\sim}{a})$, $C_{-1}(X(\underset{\sim}{a}))$ denotes the free $\mathbb{Z}$-module on $\mathcal{F}(\underset{\sim}{a},\emptyset)$ as basis and $R(\underset{\sim}{a}) = \bigoplus_{\mathcal{G}\in\mathcal{F}(\underset{\sim}{a},\emptyset)} \tilde{H}_{n-2}(X_{\mathcal{G}}(\underset{\sim}{a}),\mathbb{Z})$.

Now for each $\underset{\sim}{b}$ write $I(\underset{\sim}{b}) = \{\text{functions } f:\mathcal{F}(\underset{\sim}{b}) \to \mathbb{Z}\}$ for the (integral) permutation representation of G on the set of all flags of type $\underset{\sim}{b}$. Then for $-1 \leq r \leq n-2$ we have

$$C_r(X(\underset{\sim}{a})) \cong \bigoplus_{|J|=r+1} I(\underset{\sim}{a},J).$$

With these identifications the maps d_i $(0 \leq i \leq n-2)$ are given by :

(i) $(d_i f)(\mathcal{J}) = \sum_{\mathcal{J}' \geq \mathcal{J}} f(\mathcal{J}')$

where $f \in I(\underset{\sim}{a},J)$ with $|J|=i+1$, $\mathcal{J}\in\mathcal{F}(\underset{\sim}{a},J-\{j\})$ for some $j\in J$, and the sum is over the q flags $\mathcal{J}'\in\mathcal{F}(\underset{\sim}{a},J)$ with $\mathcal{J}' \geq \mathcal{J}$.

(ii) $(d_i f)(\mathcal{J}) = 0$ if f is as before, but $\mathcal{J}\in\mathcal{F}(\underset{\sim}{a},K)$ with $K \not\subset J$.

(iii) $d_i f$ is extended by $\mathbb{Z}$-linearity to all of $C_{i-1}(\underset{\sim}{a})$.

Note that each module in (‡) admits a natural G-action, and that these actions are compatible with the maps d_i. Also note that $R(\underset{\sim}{a}) = \ker(d_{n-2})$ and that if $n \geq 3$ then $R(\underset{\sim}{a}) = H_{n-2}(X(\underset{\sim}{a}),\mathbb{Z})$ since in this case $\tilde{H}_{n-2}(Y,\mathbb{Z}) = H_{n-2}(Y,\mathbb{Z})$, and, of course, $H_{n-2}(X(\underset{\sim}{a}),\mathbb{Z}) = \bigoplus_{\mathcal{G}\in\mathcal{F}(\underset{\sim}{a},\emptyset)} H_{n-2}(X_{\mathcal{G}}(\underset{\sim}{a}),\mathbb{Z})$.

From (‡) and the description of d_{n-2} we have :

<u>Corollary 3.23</u>

(i) $R(\underset{\sim}{a}) \cong \sum_{J\subseteq S} (-1)^{|J'|} I(\underset{\sim}{a},J)$

(ii) $R(\underset{\sim}{a}) \cong \{f:I(\underset{\sim}{a}) \to \mathbb{Z}$ satisfying c_i for $1 \leq i \leq n-1\}$,

where c_i is the <u>i'th cycle condition</u> :

for each $\mathcal{J}_i\in\mathcal{F}(\underset{\sim}{a},\{i\}')$ we have $\sum_{\mathcal{J}\geq\mathcal{J}_i} f(\mathcal{J}) = 0$, the sum being taken over the q flags $\mathcal{J}\in\mathcal{F}(\underset{\sim}{a})$ with $\mathcal{J} \geq \mathcal{J}_i$. ■

Corollary 3.24

$R(\underset{\sim}{a})$ has rank, as free $\mathbb{Z}$-module, $\displaystyle\prod_{i=1}^{n-1} \frac{q^{ih}-q^{ih-(i+1)}}{q^{h-a_i}}$.

Proof:

$$\begin{aligned}
\operatorname{rank}(R(\underset{\sim}{a})) &= \sum_{J\subset S} (-1)^{|J'|}\operatorname{rank}(I(\underset{\sim}{a},J)) && \text{by §3.23(i)}\\
&= \sum_{J\subset S} (-1)^{|J'|}|\mathcal{F}(\underset{\sim}{a},J)| \\
&= |\mathcal{F}(\underset{\sim}{a},\emptyset)| . \sum_{J\subset S} (-1)^{|J'|}q^{|J|} && \text{by §3.16(i)}\\
&= \prod_{i=1}^{n-1} \left(q^{a_i-h} . \frac{q^{ih}-q^{ih-(i+1)}}{q-1}\right).(q-1)^{n-1} && \text{by §3.16(ii)}\\
&= \prod_{i=1}^{n-1} \frac{q^{ih}-q^{ih-(i+1)}}{q^{h-a_i}}. && \blacksquare
\end{aligned}$$

Now let F be any field and denote $R(\underset{\sim}{a})_F = R(\underset{\sim}{a}) \otimes_{\mathbb{Z}} F$, etc.

Theorem 3.25

$\langle R(\underset{\sim}{a})_F \, , \, I(\underset{\sim}{a})_F \rangle_{FG} = 1.$

Corollary 3.26

$R(\underset{\sim}{a})_F$ is an irreducible FG-module.

Remarks

(i) Let $\underset{\sim}{b} \geq \underset{\sim}{a}$. Then $I(\underset{\sim}{a})$ may be identified with a subrepresentation of $I(\underset{\sim}{b})$: $I(\underset{\sim}{a}) = \{f \in I(\underset{\sim}{b})$ such that $f(\mathfrak{z}) = f(\mathfrak{z}')$ for all $\mathfrak{z},\mathfrak{z}' \in \mathcal{F}(\underset{\sim}{b})$ whose unique $\mathfrak{g},\mathfrak{g}' \in \mathcal{F}(\underset{\sim}{a})$ with $\mathfrak{g} \leq \mathfrak{z}, \mathfrak{g}' \leq \mathfrak{z}'$ are the same : $\mathfrak{g} = \mathfrak{g}'\}$. Thus all the $R(\underset{\sim}{a})_F$ are irreducible components of 1_B^G.

(ii) We shall also denote $R(h^{n-1})$ by St_G; it is a candidate for the name of 'Steinberg representation' of G as it is the 'largest' irreducible (over F) component of 1_B^G , in the sense that its degree, as a polynomial in q, has the same leading term as the degree of 1_B^G, viz. $q^{n(n-1)/2}$ (§3.16,§3.24/5).

Further justification is provided by its connection with the representation χ^H_U of the affine group (§3.4).

We remark again that our constructions are valid only for $h \geq 2$; the connection with the case $h = 1$ will be investigated in Chapter 4.

(iii) We also note that $\mathrm{rank}(St_G)$ is the same as the dimension of the fully ramified discrete series representations of G, if we use the formula $|G|/|T|.|U|$ for the latter : here T is a 'fully ramified torus', i.e. a subgroup of G isomorphic to the group of units R'^* in a 'fully ramified extension' R' of R of degree n (suitably defined : cf.§1.0), and U is the unipotent group of §1.3. (Such representations have already been constructed by Howe in the tamely ramified case [11]).

Proof of 3.25

For convenience we suppress reference to F.

By §3.17 $I(\underset{\sim}{a}) \cong 1^G_{B_{\underset{\sim}{a}}}$, so by Frobenius reciprocity

$$< R(\underset{\sim}{a}) \ , \ I(\underset{\sim}{a}) >_G = < R(\underset{\sim}{a})|_{B_{\underset{\sim}{a}}} \ , \ 1_{B_{\underset{\sim}{a}}} >_{B_{\underset{\sim}{a}}} ,$$

so it is sufficient to prove that $\mathrm{Hom}_{B_{\underset{\sim}{a}}}(R(\underset{\sim}{a})|_{B_{\underset{\sim}{a}}}, 1_{B_{\underset{\sim}{a}}}) \cong \mathcal{E}_{\underset{\sim}{a}}$ is one-dimensional, where

$$\mathcal{E}_{\underset{\sim}{a}} = \{f : \mathcal{F}(\underset{\sim}{a}) \to F \text{ such that } f \text{ satisfies } c_i \ (1 \leq i \leq n-1) \text{ and } f \text{ is constant on } B_{\underset{\sim}{a}}\text{-orbits of } \mathcal{F}(\underset{\sim}{a})\}.$$

Let $X = \{vw : w \in W, \ v \in U^-(\pi R) \cap {}^wU^-(\pi R)\}$. Then $B_{\underset{\sim}{a}}.X \supset \mathcal{B}_{\underset{\sim}{a}}$ so by §3.18 $B_{\underset{\sim}{a}} X B_{\underset{\sim}{a}} = G$. Thus if $f \in \mathcal{E}_{\underset{\sim}{a}}$ its values are determined by its values on $\{g.\mathfrak{F}_{\underset{\sim}{a}} : g \in X\} = \mathcal{F}(\underset{\sim}{a})_X$.

We shall prove that if $\mathfrak{F} \in \mathcal{F}(\underset{\sim}{a})_X$ then either $f(\mathfrak{F}) = 0$ or $f(\mathfrak{F}) = k^{-1} f(\mathfrak{F}_{\underset{\sim}{a}})$ for some non-zero integer k. This implies that $\dim(\mathcal{E}_{\underset{\sim}{a}}) \leq 1$; but $R(\underset{\sim}{a}) \subset I(\underset{\sim}{a})$ implies $\dim(\mathcal{E}_{\underset{\sim}{a}}) \geq 1$. So we shall have shown $\dim(\mathcal{E}_{\underset{\sim}{a}}) = 1$, and the theorem will be proved;

(i) First consider $\mathfrak{F} = w\mathfrak{F}_{\underset{\sim}{a}} = (0 \subset U_1 \subset \dots \subset U_{n-1} \subset V)$ where $1 \neq w \in W$. Then $U_i = \langle e_{w(1)},\dots,e_{w(i-1)},\pi^{h-a_i}.e_{w(i)} \rangle$ for $1 \leq i \leq n-1$. Choose j such that $1 \leq j \leq n-1$ and $w(j) > w(j+1)$.

Now by §3.16 there exists a unique flag $\mathfrak{F}_j =$ $(0 \subset U_1 \subset \dots \subset W_j \subset \dots \subset U_{n-1} \subset V) \in \mathfrak{F}(\underset{\sim}{a},\{j\}')$ with $\mathfrak{F} \geq \mathfrak{F}_j$ and there exist precisely q flags $\mathfrak{F}_r =$ $(0 \subset U_1 \subset \dots \subset U_r \subset \dots \subset U_{n-1} \subset V) \in \mathfrak{F}(\underset{\sim}{a})$ such that $\mathfrak{F}_r \geq \mathfrak{F}_j$ (one of which is $\mathfrak{F}$). In fact

$W_j = \langle e_{w(1)},\dots,e_{w(j-1)},\pi^{h-a_j+1}e_{w(j)} \rangle$ and

$U_r = \langle e_{w(1)},\dots,e_{w(j-1)},\pi^{h-a_j}.e_{w(j)}+\tilde{r}.\pi^{h-1}e_{w(j+1)} \rangle \quad (r \in \bar{R})$.

Now $u_{r,s} = x_{w(j+1),w(j)}((\tilde{s}-\tilde{r})\pi^{a_j-1}) \in B_{\underset{\sim}{a}}$ $(r,s \in \bar{R})$ and sends $e_{w(j)}$ to $e_{w(j)}+(\tilde{s}-\tilde{r})\pi^{a_j-1}e_{w(j+1)}$, but fixes $e_{w(i)}$ if $i \neq j$. So $u_{r,s}.\mathfrak{F}_r = \mathfrak{F}_s$ for $r,s \in \bar{R}$. So since $f \in \mathcal{E}_{\underset{\sim}{a}}$ is constant on $B_{\underset{\sim}{a}}$-orbits we have $f(\mathfrak{F}_r) = f(\mathfrak{F})$ for all $r \in \bar{R}$.

However, f also satisfies condition c_j so $\sum_{r\in\bar{R}} f(\mathfrak{F}_r) = 0$, and hence $f(\mathfrak{F}) = 0$.

(ii) Next consider $\mathfrak{G} = vw\mathfrak{F}_{\underset{\sim}{a}} = v.\mathfrak{F}$, $\mathfrak{F}$ as in (i) and $v \in U^-(\pi R)$. Now $v.\mathfrak{F}_j \leq v.\mathfrak{F} = \mathfrak{G}$, since $\mathfrak{F}_j \leq \mathfrak{F}$, and $v.\mathfrak{F}_j \in \mathfrak{F}(\underset{\sim}{a},\{j\}')$; so $v.\mathfrak{F}_j$ is the unique such flag given by §3.16. Also, for each $r \in \bar{R}$ $v.\mathfrak{F}_r \in \mathfrak{F}(\underset{\sim}{a})$ and $v.\mathfrak{F}_r \geq v.\mathfrak{F}_j$. But the flags $v.\mathfrak{F}_r (r\in\bar{R})$ are all distinct (since $v.\mathfrak{F}_r = v.\mathfrak{F}_s$ implies $v^{-1}v.\mathfrak{F}_r = v^{-1}v.\mathfrak{F}_s$ i.e. $\mathfrak{F}_r = \mathfrak{F}_s$) so they are precisely the q flags of type $\underset{\sim}{a}$ which are $\geq v.\mathfrak{F}_j$, and one of them is $\mathfrak{G}$.

Hence $\sum_{r\in\bar{R}} f(v.\mathfrak{F}_r) = 0$.

But $u_{r,s} \in U$ and $v \in U^-(\pi R)$ so by §1.34 we have $u_{r,s}v.\mathfrak{F}_r = bvu_{r,s}.\mathfrak{F}_r = bv.\mathfrak{F}_s$ for some $b\in B$ whence $f(\mathfrak{G}) = f(v.\mathfrak{F}_r)$ for all $r \in \bar{R}$ and so $f(\mathfrak{G}) = 0$.

(iii) Finally consider $\mathfrak{H} = v.\mathfrak{F}_{\underset{\sim}{a}} = (0 \subset U_1 \subset \dots \subset U_{n-1} \subset V)$ where $v \in U^-(\pi R)$.

Let $1\leq j\leq n-1$ and let $\mathcal{H}_j \in \mathcal{F}(\underset{\sim}{a},\{j\}')$ be the unique flag of its type which is $\leq \mathcal{H}$, and let $\mathcal{H}_r = (0 \subset U_1 \subset .. \subset U_r \subset .... \subset V)$ $(r \in \overline{R})$ be the q flags of type $\underset{\sim}{a}$ which are $\geq \mathcal{H}_j$.

Since $f \in \mathcal{E}_{\underset{\sim}{a}}$ we have $\sum_{r\in\overline{R}} f(\mathcal{H}_r) = 0$.

Assume that with respect to the basis $\underset{\sim}{e}$, v is the matrix (a_{ij}) with $a_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \end{cases}$.

Lemma 3.27

Assume that $U_i = V_i$ for $j+1\leq i\leq n-1$, i.e. $a_{i+1,k} = 0$ for $1\leq k\leq i, j+1\leq i\leq n-1$; but suppose $U_j \neq V_j$, i.e. $a_{j+1,j}\pi^{h-a_j} \neq 0$.

Then (i) if $\pi^{a_j-1} \nmid a_{j+1,j}$ then $f(\mathcal{H}) = 0$

(ii) if $\pi^{a_j-1} | a_{j+1,j}$ then $f(\mathcal{H}) = (1-q)^{-1}.f(\mathcal{H}_{-m})$,

where $U_{-m} = V_j$.

Proof:

$U_r = < ve_1,\dots,ve_{j-1}, \pi^{h-a_j}(e_j+a_{j+1,j}e_{j+1}) + \tilde{r}\pi^{h-1}e_{j+1} > \quad (r\in\overline{R})$.

(i) Write $a_{j+1,j} = \tilde{m}\pi^{l}$ where $m \in \overline{R}^*$; then $l < a_j-1$, so
$\pi^{h-a_j}a_{j+1,j}+\tilde{r}\pi^{h-1} = t_r\pi^{h-a_j+l}$, where $t_r = \tilde{m}+\tilde{r}\pi^{a_j-1-l} \in R^*$.

Then for $r,s \in \overline{R}$,

$$t_{r,s} = \begin{pmatrix} I_j & & 0 \\ & t_s t_r^{-1} & \\ 0 & & I_{n-1-j} \end{pmatrix} \in B_{\underset{\sim}{a}} \text{ and fixes}$$

e_i for $i\neq j+1$, and sends $(\pi^{h-a_j}a_{j+1,j}+\tilde{r}\pi^{h-1})e_{j+1}$ to

$(\pi^{h-a_j}a_{j+1,j}+\tilde{s}\pi^{h-1})e_{j+1}$; it also fixes ve_i if $i\neq j+1$ since it commutes with v (v acts trivially on e_{j+1}, the only basis vector where $t_{r,s}$ is non-trivial).

Thus $t_{r,s}.U_i = U_i$ if $i\neq j$ and $t_{r,s}U_r = U_s$ $(r,s \in \overline{R})$, i.e. $t_{r,s}.\mathcal{H}_r = \mathcal{H}_s$. Hence $f(\mathcal{H}) = f(\mathcal{H}_r)$ for all $r\in\overline{R}$ and $f(\mathcal{H}) = 0$.

(ii) $\pi^{a_j-1} | a_{j+1,j}$ and $\pi^{h-a_j}a_{j+1,j}\neq 0$ implies that
$\pi^{h-a_j}a_{j+1,j} = \tilde{m}\pi^{h-1}$ where $m \in \overline{R}^*$.

Assume that $r,s \in \overline{R}$ with $r \neq -m$, $s \neq -m$. Then

$$t'_{r,s} = \begin{pmatrix} I_j & & 0 \\ & \frac{\tilde{s}+\tilde{m}}{\tilde{r}+\tilde{m}} & \\ 0 & & I_{n-j-1} \end{pmatrix} \in B_{\underset{\sim}{a}} \text{ and fixes } e_i \text{ if } i \neq j+1, \text{ and}$$

sends $\pi^{h-a_j}(e_j + a_{j+1,j}e_{j+1}) + \tilde{r}\pi^{h-1}e_{j+1}$ to

$\pi^{h-a_j}(e_j + a_{j+1,j}e_{j+1}) + \tilde{s}\pi^{h-1}e_{j+1}$; it also fixes ve_i if $i \neq j+1$

since it commutes with v.

Hence $t'_{r,s}.\mathcal{H}_r = \mathcal{H}_s$.

Now $U_{-m} = V_j$ so $\mathcal{H} = \mathcal{H}_r$ for some $r \neq -m$, whence $f(\mathcal{H}) = f(\mathcal{H}_r)$ for any $r \in \overline{R}$ with $r \neq -m$. Thus we have $f(\mathcal{H}).(q-1) + f(\mathcal{H}_{-m}) = 0$, and the lemma is proved. ∎

It follows from the lemma, by decreasing induction on j, that if $\mathcal{H} \neq \mathfrak{J}_{\underset{\sim}{a}}$ then either $f(\mathcal{H}) = 0$ or else there exists $k \in \mathbb{Z}$ with $k \neq 0$ such that $f(\mathcal{H}) = k^{-1}f(\mathfrak{J}_{\underset{\sim}{a}})$.

Together with (i) and (ii) this completes the proof of §3.25. ∎

We now prove a vanishing theorem for the character of $R(\underset{\sim}{a})$ on certain 'regular enough' semisimple elements.

<u>Theorem 3.28</u>

Suppose $\underset{\sim}{a} = (a_1,\ldots,a_{n-1})$ is an admissible sequence and write $m = \min\{a_i : 1 \leq i \leq n-1\}$. Let t be an unramified semisimple element of G.

(i) if t is split and $k_\alpha(t) \leq m-2$ for all $\alpha \in \Phi$ then $R(\underset{\sim}{a})(t)=0$.

(ii) if t is regular then $R(\underset{\sim}{a})(t) = 0$.

(note that $\underset{\sim}{a}$ admissible implies $m \geq 2$).

Proof:

First note that by §3.23(i)

$$R(\underset{\sim}{a})(t) = \sum_{J\subseteq S} (-1)^{|J'|}.|\{\text{flags of type } (\underset{\sim}{a},J) \text{ fixed by } t\}|.$$

(i) If $\mathfrak{F}\in\mathcal{F}(\underset{\sim}{a},J)$ then it may be written uniquely in the form $c.\mathfrak{F}_{(\underset{\sim}{a},J)}$ where $c = uw \in \mathcal{B}_{(\underset{\sim}{a},J)}$ (§3.18); but tc may be written uniquely in the form $tc = c'.b'$ with $c' \in \mathcal{B}_{(\underset{\sim}{a},J)}$ and $b' \in B_{(\underset{\sim}{a},J)}$, so $t.\mathfrak{F} = \mathfrak{F}$ iff $c' = c$.

We have $tc = tuw = t.\prod_{\alpha<0} x_{w\alpha}(r_\alpha).w$, say

$$= \prod_{\alpha<0} (t.x_{w\alpha}(r_\alpha).t^{-1}).w.w^{-1}tw$$

$$= \prod_{\alpha<0} (x_{w\alpha}((w\alpha)(t).r_\alpha)).w.w^{-1}tw \text{ , since}$$

T normalises each $U_{w\alpha}$. This may be written as $(\prod_{\alpha<0} x_{w\alpha}(r'_\alpha)).w.(\prod_{-\alpha\in\Sigma} x_\alpha(s_\alpha).w^{-1}tw)$, where $r'_\alpha = (w\alpha)(t).r_\alpha$ if $-\alpha\notin\Sigma$ and if $-\alpha = \alpha_{i,i+1} \in\Sigma$ then $(w\alpha)(t).r_\alpha = r'_\alpha + s_\alpha$ with

$$r'_\alpha \in \begin{cases} R_{a_i} & \text{if } i\in J \\ R_{a_i-1} & \text{if } i\notin J \end{cases} \quad \text{and} \quad s_\alpha \in \begin{cases} \pi^{a_i}R & \text{if } i\in J \\ \pi^{a_i-1}R & \text{if } i\notin J \end{cases};$$

(note that if $\pi|r_\alpha$ then $\pi|r'_\alpha$).

Thus $c' = \prod_{\alpha<0} x_{w\alpha}(r'_\alpha).w \in \mathcal{B}_{(a,J)}$, $b' = \prod_{-\alpha\in\Sigma} x_\alpha(s_\alpha).w^{-1}tw \in B_{\underset{\sim}{a}}$; and so $c' = c$ iff $r'_\alpha = r_\alpha$ for all $\alpha\in\Phi^-$.

Now if $\alpha = -\alpha_{i,i+1}$ and $(w\alpha)(t) = 1 + m\pi^k$ with $m \in R^*$, then $r_\alpha = r'_\alpha$ iff $m\pi^k r_\alpha = s_\alpha$ and the number of solutions of this equation depends on whether or not $i \in J$ iff $k \geq a_i-1$. On the other hand, if $-\alpha\notin\Sigma$ then the number of solutions to $r_\alpha = r'_\alpha = (w\alpha)(t).r_\alpha$ is clearly independent of J. The point now is that the hypothesis $k_\alpha(t) \leq m-2$ implies that $k_{w\alpha}(t) < a_i-1$ for $\alpha\in\Phi, w \in W$ and <u>all</u> $i \in S$, so the number of flags of type $(\underset{\sim}{a},J)$ fixed by t is independent of J; in fact the number would be the same replacing $(\underset{\sim}{a},J)$ by any $\underset{\sim}{b} \geq (\underset{\sim}{a},\emptyset)$.

Hence $R(\underset{\sim}{a})(t) = 0$, by the binomial theorem.

(ii) First note that t regular implies $\overline{t}$ regular (§1.58) and also $k_\alpha(t) = 0$ for all $\alpha \in \Phi$ (§1.59).

We may assume t is not split, since part (i) covers that case. Then $\overline{t}$ is not split, for if it were then there would exist distinct eigenvalues $t_i = \xi, t_j = F^1\xi$ of t, for which $\pi_1(t_i) = \pi_1(t_j)$ whence $\pi | (t_i - t_j)$, contrary to $k_\alpha(t) = 0$.

Hence no conjugate of $\overline{t}$ lies in $\overline{B} = \overline{B}_{(\underset{\sim}{a},J)}$, and so no conjugate of t can lie in $B_{(\underset{\sim}{a},J)}$. Thus $I(\underset{\sim}{a},J)(t) = 0$ for all $J \subset S$, and so $R(\underset{\sim}{a})(t) = 0$. ∎

<u>Remark</u>

We have been able to improve §3.28(i) in the cases n=2,3 with $\underset{\sim}{a} = (h^{n-1})$, weakening the hypothesis to $k_\alpha(t) \leq m-1$ for all $\alpha \in \Phi$; but in this case the number of flags of type $(\underset{\sim}{a},J)$ fixed by t is not independent of J. Further details will be given in Chapter 5.

<u>3.3 Characters of U</u>

The commutator subgroup U' of U is $\prod_{1 \leq i < j-1 \leq n-1} U_{ij}$ (§1.35).
It follows that the set $\hat{U}$ of 1-dimensional characters of U (homomorphisms $\chi : U \to \mathbb{C}^*$) may be identified with the character group of $U^{ab} = \prod_{i=1}^{n-1} U_{i,i+1} \cong U/U'$. Thus if $u = \prod_{\alpha > 0} x_\alpha(r_\alpha) \in U$ then $\chi(u) = \chi(\prod_{i=1}^{n-1} x_{i,i+1}(r_{i,i+1}))$ for $\chi \in \hat{U}$.

Fix a <u>non-singular</u> additive character ψ of R, i.e. $\psi | \pi^{h-1}R \neq 1$. For $1 \leq k \leq n-1$ define $\chi_k \in \hat{U}$ by $\chi_k(\prod_{i=1}^{n-1} x_{i,i+1}(s_i)) = \psi(s_k)$, and for $r_k, s_i \in R$ define

$$(\sum_{k=1}^{n-1} r_k \chi_k).(\prod_{i=1}^{n-1} x_{i,i+1}(s_i)) = \psi(\sum_{j=1}^{n-1} r_j s_j).$$

Proposition 3.31

We have an isomorphism $\theta : U^{ab} \to \hat{U}$ given by

$$\prod_{i=1}^{n-1} x_{i,i+1}(r_i) \mapsto \sum_{i=1}^{n-1} r_i \chi_i.$$

Proof:

The isomorphisms $x_{i,i+1}$ $(1 \leq i \leq n-1)$ provide an isomorphism of U^{ab} with R^{n-1} (the direct product of n-1 copies of R^+), and we shall henceforth identify $U^{ab} = R^{n-1}$ and abuse notation by writing $\prod_{i=1}^{n-1} x_{i,i+1}(r_i) = (r_1,\ldots,r_{n-1})$.

Then θ is clearly homomorphic. θ is injective since if $\theta(r_1,\ldots,r_{n-1}) = \theta(r_1',\ldots,r_{n-1}')$ then $\psi(\sum_{i=1}^{n-1} r_i s_i) = \psi(\sum_{i=1}^{n-1} r_i' s_i)$ for all $(s_1,\ldots,s_{n-1}) \in U^{ab}$ and hence $\psi(r_i) = \psi(r_i')$ $(1 \leq i \leq n-1)$, whence $r_i = r_i'$ $(1 \leq i \leq n-1)$ since ψ is non-singular. Finally, θ is surjective since U^{ab} is abelian and so $|U^{ab}| = |\hat{U}^{ab}| = |\hat{U}|$. ∎

Definition

Let $\chi = \sum_{i=1}^{n-1} r_i \chi_i \in \hat{U}$. We say that χ is non-singular or in general position iff $r_i \in R^*$ $(1 \leq i \leq n-1)$.

Now T acts on each U_α $(\alpha \in \Phi)$ by conjugation, and so T acts on $U^{ab} = R^{n-1}$ by conjugation : in fact, since

$$t.\prod_{i=1}^{n-1} x_{i,i+1}(r_i) t^{-1} = \prod_{i=1}^{n-1}(t x_{i,i+1}(r_i) t^{-1}) = \prod_{i=1}^{n-1} x_{i,i+1}(\alpha_{i,i+1}(t).r_i)$$

we see T acts on $U^{ab} = R^{n-1}$ by

$$^t(r_1,\ldots,r_{n-1}) = (\alpha_{12}(t).r_1,\ldots,\alpha_{n-1,n}(t).r_{n-1}).$$

Proposition 3.32

T acts on $\hat{U}$ and the set of non-singular $\chi \in \hat{U}$ form one complete orbit under the T-action.

Proof:

As usual, T acts on $\hat{U}$ by $(t.\chi)(u) = \chi(tut^{-1})$ $(u \in U)$. Writing $u = (u_1,\ldots,u_{n-1})$, $\chi = \sum_{i=1}^{n-1} r_i \chi_i$ we have

$$(t.\chi)(u) = \chi(\alpha_{12}(t)u_1,\dots,\alpha_{n-1,n}(t)u_{n-1})$$
$$= (\sum_{i=1}^{n-1} r_i\alpha_{i,i+1}(t).\chi_i)(u_1,\dots,u_{n-1})$$

i.e. $t.\chi = \sum_{i=1}^{n-1} r_i\alpha_{i,i+1}(t).\chi_i$.

So if χ is non-singular, i.e. $r_i \in R^*$ $(1\leq i\leq n-1)$ then $r_i\alpha_{i,i+1}(t) \in R^*$ $(1\leq i\leq n-1)$ so $t.\chi$ is non-singular.

Moreover, T is transitive on the set of $\chi \in \hat{U}$ which are non-singular, since given $r_i, r_i' \in R^*$ $(1\leq i\leq n-1)$ define $t_1=1$, $t_i = t_{i+1}r_i'r_i^{-1}$ $(1\leq i\leq n-1)$ and define $t \in G$ by $t(e_i) = t_ie_i$ $(1\leq i\leq n)$; then $t \in T$ and sends $\sum_{i=1}^{n-1} r_i\chi_i$ to $\sum_{i=1}^{n-1} r_i'\chi_i$. ∎

Now with respect to the basis $\underset{\sim}{e}$ of V, G consists of all non-singular $n\times n$ matrices, and $U = \{(a_{ij}) \in G : a_{ij} = 0$ if $i > j$, and $a_{ii} = 1$ $(1\leq i\leq n)\}$. Define the following subgroups of G for $1\leq r\leq n$:

$$G_r = \{ \begin{pmatrix} M & 0 \\ 0 & I_{n-r} \end{pmatrix} : M \text{ a non-singular } r\times r \text{ matrix}\},$$

$$H_r = \{ \begin{pmatrix} M & 0 \\ 0 & I_{n-r} \end{pmatrix} \in G_r : M = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \text{ with } A \text{ a non-singular}$$
$$(r-1)\times(r-1) \text{ matrix and } v \text{ a column vector } \in R^{r-1}\}.$$

We have $G_{r-1} \subset H_r \subset G_r$ $(1\leq r\leq n)$, where we write $G_0 = 1$, and also $U \subset H = H_n$.

<u>Theorem 3.33</u>

Let χ be a non-singular character of U. Then the induced representation χ_U^H is irreducible.

<u>Proof</u>:

By Frobenius reciprocity $\langle\chi_U^H, \chi_U^H\rangle_H = \langle\chi_U^H|_U, \chi\rangle_U$, so as usual $\mathcal{E} = \mathrm{End}_H(\chi_U^H) \cong \{f:H \to \mathbb{C}$ such that $f(ux) = f(xu) = \chi(u)f(x)$ for all $u \in U$, $x \in H\}$, and we are to prove that this algebra is one-dimensional.

Now if $f \in \mathcal{E}$ and $x \in H$ then $f(x) \neq 0$ iff $f(y) \neq 0$ for all the elements $y = u_1 x u_2$ of the U-U double coset of H containing x, since $f(y) = \chi(u_1 u_2) f(x)$. An arbitrary element $h \in H$ may be written in the form $h = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & v \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, and by §1.46 $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-1} \cong GL_{n-1}(R)$ may be written in the form $u_1 t_1 z w t_2 u_2$ where $u_1, u_2 \in U$, $t_1, t_2 \in T_{n-1} = \{(a_{ij}) \in G : a_{ij} = 0$ if $i \neq j$, $a_{nn} = 1\}$, $w \in W$, and $z \in U^-(\pi R)$. So the values of f are determined on elements of the form $t_1 w z t_2$.

Also $\begin{pmatrix} I_{n-1} & v \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} I_{n-1} & A^{-1}v \\ 0 & 1 \end{pmatrix}$,

so $\chi\begin{pmatrix} I_{n-1} & v \\ 0 & 1 \end{pmatrix} . f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} . \chi\begin{pmatrix} I_{n-1} & A^{-1}v \\ 0 & 1 \end{pmatrix}$,

i.e. $\left(\chi\begin{pmatrix} I_{n-1} & A^{-1}v - v \\ 0 & 1 \end{pmatrix} - 1\right) . f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = 0$ ----(*)

<u>Lemma 3.34</u>

Let $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = t_1 z w t_2$ as above and suppose that the $(n-1)$'th coordinate $(A^{-1}v - v)_{n-1}$ of the column vector $(A^{-1}v - v)$ is zero for all column vectors $v \in R^{n-1}$. Then $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-2}$.

<u>Proof</u>:

Equivalently, we may prove :

(**) : If $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \notin G_{n-2}$ then there exists $v \in R^{n-1}$ such that $(A^{-1}v - v)_{n-1} \neq 0$.

Now A^{-1} has the form $t_2^{-1} w^{-1} z^{-1} t_1^{-1}$; we note $w^{-1} \in W$, t_1^{-1} and $t_2^{-1} \in T_{n-1}$, $z^{-1} \in U^-(\pi R)$. So the l'th entry of $A^{-1}v$ has the form $(A^{-1}v)_l = \sum_{i=1}^{j} a_{ij} \pi^{m_{ij}} v_i$, where $v_i = (v)_i$, $l = w^{-1}(j)$, $a_{ij} \in R^*$, $1 \leq m_{ij} \leq h$ $(i<j)$ and $m_{jj} = 0$ (and $1 \leq l \leq n-1$).

<u>Case 1</u> : $w^{-1}(n-1) \neq n-1$.

Then $\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \notin G_{n-2}$, so $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \notin G_{n-2}$. But we may choose $v_1 = \dots = v_{n-2} = 0$ and $v_{n-1} \neq 0$. So $(A^{-1}v - v)_{n-1} = -v_{n-1} \neq 0$.

<u>Case 2</u> : $w^{-1}(n-1) = n-1$. There are three subcases :

(i) if there exists k such that $1 \leq k \leq n-2$ and $m_{k,n-1} \neq h$ then choose $v_i = 0$ ($1 \leq i \leq n-1$, $i \neq k$), $v_k \in R^*$. Then $(A^{-1}v-v)_{n-1} = a_{k,n-1}\pi^{m_{k,n-1}}v_k \neq 0$.

(ii) if $m_{k,n-1} = h$ ($1 \leq k \leq n-2$) and $a_{n-1,n-1} \neq 1$ then choose $v_{n-1} \in R^*$. Then $(A^{-1}v-v)_{n-1} = (a_{n-1,n-1}-1)v_{n-1} \neq 0$.

(iii) if $m_{k,n-1} = h$ ($1 \leq k \leq n-2$) and $a_{n-1,n-1} = 1$ then $\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-2}$, so $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-2}$ and there is nothing to prove.

The lemma is now proved. ■

Returning to the proof of the theorem, (*) shows that if $f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ then $\chi\begin{pmatrix} I_{n-1} & A^{-1}v-v \\ 0 & 1 \end{pmatrix} = 1$, so since χ is non-singular $(A^{-1}v-v)_{n-1} = 0$, and so by the lemma $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-2}$.

But now the same proof as before (but with n-1 instead of n) shows that since $f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ we have $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in G_{n-3}$ (if $n \geq 3$); and an evident induction shows that if $f\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ then $A = I_{n-1}$. Hence f takes non-zero values only on the elements of the U-U double coset of H containing 1, and $\mathcal{E}$ is one-dimensional. Theorem 3.33 is proved. ■

Remark: it follows from §3.32, since $T \subset H$, that χ_U^H is independent of the particular non-singular $\chi \in \hat{U}$ we choose.

3.4 The affine Steinberg representation

An affine module of type $\underline{a}$ in V is a coset $x + W$ where $x \in V$ and W is a submodule of V of type $\underline{a}$. Affine modules A, A' (in V) are parallel iff there exists $y \in V$ such that either $A + y \subset A'$ or $A + y \supset A'$.

An affine module $A = x + W$ is in general position if there exists $x \in A$ such that $\pi^{h-1}x \notin W$. We note that this condition is then fulfilled by any $y \in A$ ($A=y+W$ for any $y\in A$, and $y=x+w$ for some $w\in W$; then if $\pi^{h-1}y\in W$ we have $\pi^{h-1}x=\pi^{h-1}y-\pi^{h-1}w \in W$, contrary to assumption). Further note that if A is in general position then any $y\in A$ is a primitive vector (i.e. $\pi^{h-1}y\neq 0$).

Now let us fix an affine hyperplane $A=x+W$ in V, i.e. an affine module of type (h^{n-1}). For each subset $J \subset S = \{1,..,n-1\}$ we define affine flags of type J in A as follows :

(we assume $h \geq 2$ for this section)

(1) In case $1\in J$ an affine flag of type J in A will be a flag $\mathcal{A} = (A_0 \subset .. \subset A_{n-2} \subset A)$ of affine modules A_i in V, subject to the conditions :

(i) A_0 has type 0, i.e. $A_0=\{v_0\}$ for some $v_0\in A$.

(ii) A_i has type $\begin{cases} (h^i) & \text{if } i+1 \in J \\ (h^{i-1},h-1) & \text{if } i+1 \notin J \end{cases}$ $(1\leq i\leq n-2)$.

(iii) for $1\leq i\leq n-2$ there exists B_i of type (h^i) such that

$$A_i \subset B_i \subset A_{i+1} \quad (\text{where } A_{n-1} = A).$$

(2) In case $1\notin J$ then by slight abuse of language an affine flag of type J will be an equivalence class of affine flags of type $J\cup\{1\}$; $\mathcal{A} = (A_0\subset .. \subset A_{n-2}\subset A)$, $\mathcal{A}' = (A_0'\subset .. \subset A_{n-2}'\subset A)$ of type $J\cup\{1\}$ are equivalent iff $A_i = A_i'$ $(1\leq i\leq n-2)$ and $\pi A_0 = \pi A_0'$,(i.e. $\pi v_0=\pi v_0'$). There are precisely q flags of type $J\cup\{1\}$ in each equivalence class.

The set of affine flags of type J will be denoted $\mathcal{F}(J)$. We also have the following alternative descriptions.

(1) In case $1\in J$ we may regard $\mathcal{A} \in \mathcal{F}(J)$ as a pair $(v_0,\mathcal{V})$ where $\mathcal{V}= (0 \subset V_1\subset .. \subset V_{n-2}\subset W)$ is a flag in W given by $A_i = v_0+V_i$ $(1\leq i\leq n-1)$ $(W = V_{n-1})$.

Thus $\mathfrak{F}$ is the (unique) flag through the origin parallel to $\mathcal{A}$, in the sense that $A_i \parallel V_i$ $(1 \leq i \leq n-1)$ (and $\mathcal{A}$ is the flag through v_0 parallel to $\mathfrak{F}$).

(2) In case $1 \notin J$ we may regard $\mathcal{A} \in \mathfrak{A}(J)$ as an equivalence class of pairs $(v_0, \mathfrak{F})$ of type $J \cup \{1\}$; $(v_0, \mathfrak{F})$ and $(v_0', \mathfrak{F}')$ are equivalent iff $\pi v_0 = \pi v_0'$ and $v_0 + \mathfrak{F} = v_0' + \mathfrak{F}'$ in the obvious notation (note this implies $\mathfrak{F} = \mathfrak{F}'$). Thus we may also regard $\mathcal{A}$ as a pair $(x_0, \mathfrak{F})$ where x_0 is an equivalence class of points in A; $v, v' \in A$ are equivalent iff $v - v' \in V_1$ and $\pi v = \pi v'$.

We may partially order the affine flags in A as follows. Let $\mathcal{A} = (v_0, \mathfrak{F}) \in \mathfrak{A}(J)$, $\mathcal{A}' = (v_0', \mathfrak{F}') \in \mathfrak{A}(J')$; then $\mathcal{A} \geq \mathcal{A}'$ iff $\mathfrak{F} \geq \mathfrak{F}'$ (as in §3.1) and either $v_0 = v_0'$ or $v_0 \in v_0'$ (as appropriate). Note that if $\mathcal{A} \geq \mathcal{A}'$ then $J' \subset J$.

<u>Lemma 3.41</u>

(i) Suppose $\mathcal{A} \in \mathfrak{A}(J)$ with $J \neq \emptyset$. Then for each $i \in J$ there exists a unique flag $\mathcal{A}_i \in \mathfrak{A}(J - \{i\})$ with $\mathcal{A} \geq \mathcal{A}_i$.

(ii) Suppose $i \in J \subset S$ and $\mathcal{A}_i \in \mathfrak{A}(J - \{i\})$. Then there exist precisely q flags $\mathcal{A} \in \mathfrak{A}(J)$ such that $\mathcal{A} \geq \mathcal{A}_i$.

<u>**Proof**</u>:

(i) In case $i \neq 1$ this follows from §3.15(i). In case $i = 1$ then if $\mathcal{A} = (v_0, \mathfrak{F})$ and $\mathcal{A}_1 = (x, \mathfrak{F}_1) \in \mathfrak{A}(J - \{1\})$ with $\mathcal{A} \geq \mathcal{A}_1$ then $\mathfrak{F} = \mathfrak{F}_1$ by §3.15(i), and either $v_0 = x$ or $v_0 \in x$; in fact $v_0 \in x$ since $\mathcal{A}_1$ has type $J - \{1\}$, so $\mathcal{A}_1$ is uniquely determined : x is the equivalence class of v_0.

(ii) In case $i \neq 1$ this follows from §3.15(ii). In case $i = 1$ then, in the notation of (i), we again see $\mathfrak{F} = \mathfrak{F}_1$, and $v_0 \in x$; this time x is given and there are precisely q elements $v_0 \in A$ which also lie in x. ∎

Corollary 3.42

Let $J \subset K \subset S$.

(i) For each $\mathcal{A} \in \mathcal{F}(K)$ there exists a unique $\mathcal{A}' \in \mathcal{F}(J)$ with $\mathcal{A} \geq \mathcal{A}'$; and for each $\mathcal{A}' \in \mathcal{F}(J)$ there exist precisely $q^{|K|-|J|}$ flags $\mathcal{A} \in \mathcal{F}(K)$ with $\mathcal{A} \geq \mathcal{A}'$.

(ii) $$|\mathcal{F}(J)| = q^{|J|} . \prod_{i=1}^{n-1} \frac{q^{ih}-q^{i(h-1)}}{q-1} .$$

Proof:

(i) is immediate from §3.41.

(ii) $$\begin{aligned} |\mathcal{F}(J)| &= q^{|J|-|S|} . |\mathcal{F}(S)| \qquad \text{by (i)} \\ &= q^{|J|-(n-1)} . |A| . |\{\text{flags of type } (h^{n-2}) \text{ in } W\}| \\ &= q^{|J|-(n-1)} . q^{(n-1)h} . q^{n-2} . \prod_{i=1}^{n-2} \frac{q^{ih}-q^{ih-(i+1)}}{q-1} \\ &\qquad \text{(using §3.16(ii))} \\ &= q^{|J|} . q^{(n-2)h+(h-1)} . \prod_{j=2}^{n-1} \frac{q^{jh-h}-q^{j(h-1)-h}}{q-1} \\ &= q^{|J|} . \prod_{j=1}^{n-1} \frac{q^{jh}-q^{j(h-1)}}{q-1} . \end{aligned}$$ ∎

We now construct a simplicial complex $X(A)$ as follows.

(i) For $0 \leq r \leq n-2$ the r-simplices of $X(A)$ are affine flags of type J in A with $|J| = r+1$.

(ii) For $0 \leq r \leq s \leq n-2$ the r-simplex $\mathcal{A}$ is a face of the s-simplex $\mathcal{A}'$ iff $\mathcal{A} \leq \mathcal{A}'$ (in the sense of partial order on flags as above).

Then using §3.42(i) and proceeding exactly as in §3.21 we find that this construction makes sense and moreover we have :

Proposition 3.43

$X(A)$ is the disjoint union $\bigcup_{\mathcal{C} \in \mathcal{F}(\emptyset)} X_{\mathcal{C}}(A)$ of $|\mathcal{F}(\emptyset)|$ copies $X_{\mathcal{C}}(A)$ of the join of n-1 copies of a discrete set of q points. ∎

Proceeding further as in §3.2 we have an exact sequence :

$$0 \to \ker(d_{n-2}) \to C_{n-2}(X(A)) \to \ldots \to C_0(X(A)) \to C_{-1}(X(A)) \to 0,$$

arising in the same way as (‡) in §3.2.

Let H be the group of affine isomorphisms $A \cong A$ (the subgroup of G which stabilises A); we may identify H with the group H_n of §3.3 by taking $A = e_n + \langle e_1, \ldots, e_{n-1} \rangle$. For each $J \subset S$ we write $I(J)$ for the (integral) permutation representation of H on the set $\mathcal{A}(J)$. Then $C_i(X(A)) \cong \bigoplus_{|J|=i+1} I(J)$ $(-1 \leq i \leq n-2)$ and each of these modules admits a natural H-action. These actions are compatible with the boundary maps.

The representation $\ker(d_{n-2})$ of H will also be denoted St_H and called the 'affine Steinberg representation'.

We have the following result analogous to §3.23.

<u>Corollary 3.44</u>

(i) $St_H \cong \sum_{J \subset S} (-1)^{|J'|} I(J)$

(ii) $St_H = \{f : I(S) \to \mathbb{Z}$ satisfying γ_i for $1 \leq i \leq n-1\}$

where γ_i is the i'th cycle condition :

for each $\mathcal{A}_i \in \mathcal{A}(\{i\}')$ we have $\sum_{\mathcal{A} \geq \mathcal{A}_i} f(\mathcal{A}) = 0$, the sum being taken over the q flags $\mathcal{A} \in \mathcal{A}(S)$ with $\mathcal{A} \geq \mathcal{A}_i$. ■

<u>Corollary 3.45</u>

St_H has rank, as free $\mathbb{Z}$-module, $\prod_{i=1}^{n-1} (q^{ih} - q^{i(h-1)})$.

<u>Proof</u>:

$$\begin{aligned}
\operatorname{rank}(St_H) &= \sum_{J \subset S} (-1)^{|J'|} \operatorname{rank}(I(J)) && \text{by §3.44(i)} \\
&= \sum_{J \subset S} (-1)^{|J'|} |\mathcal{A}(J)| \\
&= \sum_{J \subset S} (-1)^{|J'|} . q^{|J|} . \prod_{i=1}^{n-1} \frac{q^{ih} - q^{i(h-1)}}{q-1} && \text{by §3.42(ii)} \\
&= \prod_{i=1}^{n-1} (q^{ih} - q^{i(h-1)}).
\end{aligned}$$ ■

We now propose to relate St_H to St_G by showing that St_H is contained in the restriction of St_G to H (§3.48). We shall do this by obtaining a suitable alternative description of St_H (§3.47). We begin with :

Definition

Let $J \subset S$ and suppose $\mathfrak{F} = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$ is a flag of type $(a_1, \ldots, a_{n-1})$ in V with $a_i = \begin{Bmatrix} h & \text{if } i \in J \\ h-1 & \text{if } i \notin J \end{Bmatrix}$. Call this 'type J' for brevity, in this section.

We shall say $\mathfrak{F}$ meets A iff $\begin{Bmatrix} A \cap V_1 \neq \emptyset & (1 \in J) \\ \pi A \cap V_1 \neq \emptyset & (1 \notin J) \end{Bmatrix}$.

Lemma 3.46

Suppose $\mathfrak{F}$ meets A. Then

(i) $\begin{Bmatrix} A \cap V_1 & (1 \in J) \\ \pi A \cap V_1 & (1 \notin J) \end{Bmatrix}$ consists of a single point v.

(ii) If $1 \notin J$ then there exists $v_1 \in V_2 \cap A$ such that $\pi v_1 = v$; and if v_2 is another such point then $v_1 - v_2 \in V_1$.

So write $v_0 = \begin{Bmatrix} v & \text{if } 1 \in J \\ v_1 & \text{if } 1 \notin J \end{Bmatrix}$.

(iii) $A \cap V_{i+1} = v_0 + (V_{i+1} \cap W)$ and has type (h^i) if $i+1 \in J$ or $i = n-1$, and has type $(h^{i-1}, h-1)$ if $i+1 \notin J$.

Proof:

(i) Since $\mathfrak{F}$ meets A there exists at least one point v in $\begin{Bmatrix} A \cap V_1 & (1 \in J) \\ \pi A \cap V_1 & (1 \notin J) \end{Bmatrix}$ and since A is in general position we have $\begin{Bmatrix} \pi^{h-1} v \neq 0 & (1 \in J) \\ \pi^{h-2} v \neq 0 & (1 \notin J) \end{Bmatrix}$, so $V_1 = \langle v \rangle$. Also, if $x \in A$ then $A = x + W$ and there exists $w \in W$ such that $v = \begin{Bmatrix} x+w & (1 \in J) \\ \pi x + w & (1 \notin J) \end{Bmatrix}$.

Now if v' is another such point then there exists $r \in R$, $w' \in W$ such that $\begin{Bmatrix} x+w' = v' = rv = r(x+w) & (1 \in J) \\ \pi x + w' = v' = rv = r(\pi x + w) & (1 \notin J) \end{Bmatrix}$. Hence $\begin{Bmatrix} (1-r)x = rw - w' & (1 \in J) \\ (1-r)\pi x = rw - w' & (1 \notin J) \end{Bmatrix} \in W$; but $x \in A$ so $\pi^{h-1} x \notin W$, and so $\begin{Bmatrix} (1-r) = 0 & (1 \in J) \\ (1-r)\pi = 0 & (1 \notin J) \end{Bmatrix}$ and hence $v = rv = v'$ in either case.

(ii) Since $1 \notin J$ $V_1 \sim (h-1)$; choose $W_1 \sim (h)$ with $V_1 \subset W_1 \subset V_2$ (such exists from the definition of flags of type J).

Let $w_1 \in W_1$ be such that $\pi w_1 = v$ (possible since $\pi W_1 = V_1$) and let $a \in A$ be such that $\pi a = v$. Then $\pi^{h-1} w_1 = \pi^{h-1} a \notin W$, so $W_1 \cap W = 0$ and hence $V = W_1 \oplus W$. Hence W_1 meets the coset A of W, and so also $\emptyset \neq \pi(W_1 \cap A) \subset \pi W_1 \cap \pi A = V_1 \cap \pi A = \{v\}$, whence $\pi(W_1 \cap A) = \{v\}$. Hence there exists $v_1 \in W_1 \cap A$ with $\pi v_1 = v$; since $W_1 \subset V_2$ the first part is proved.

For the rest we note that if $\pi v_1 = \pi v_2$ then $\pi^{h-1} \mid (v_1 - v_2)$ and so $v_1 - v_2 \in \pi^{h-1} W_1 \subset \pi W_1 = V_1$.

(iii) If $1 \in J$ write $W_1 = V_1$; then in any case $W_1 = \langle v_0 \rangle$ and $V = W \oplus W_1$. Now $W_1 \subset V_{i+1}$ if $1 \leq i \leq n-1$, so considering the projection $p: W \oplus W_1 \to W$, which has kernel W_1, we see $V_{i+1} \cap W = p(V_{i+1}) \cong V_{i+1}/W_1$. The assertion about the type of $V_{i+1} \cap W$ now follows from the facts that $W_1 \sim (h)$ and $\mathfrak{F}$ has type J. Finally, if $i \geq 1$ and $w \in W$ then $v_0 + w \in V_{i+1}$ iff $w \in V_{i+1} \cap W$ (since $v_0 \in V_2 \subset V_{i+1}$), so because $A = v_0 + W$ we have $A \cap V_{i+1} = v_0 + (V_{i+1} \cap W)$. ∎

<u>Notation</u> : if $\mathfrak{F}$ meets A then we define $\mathfrak{F} \cap A$ as follows.
If $1 \in J$ then put $\mathfrak{F} \cap A = (A_0 \subset A_1 \subset \ldots \subset A_{n-2} \subset A)$, where $A_i = A \cap V_{i+1}$ $(1 \leq i \leq n-1)$ and $A_0 = \{v_0\}$.
If $1 \notin J$ then we note by §3.46(ii) that the q flags $\mathfrak{F}' \cap A$ $(\mathfrak{F}' \in \mathfrak{A}(J \cup \{1\}), \mathfrak{F}' \geq \mathfrak{F})$ form an equivalence class; we denote this class by $\mathfrak{F} \cap A$.

Also define $\mathfrak{M}(J) = \{$flags of type J in V which meet A$\}$ $(J \subset S)$.
We shall say that a function $f: \mathfrak{M}(S) \to \mathbb{Z}$ satisfies the i'th cycle condition c_i' iff for each $\mathfrak{F}_i \in \mathfrak{M}(\{i\}')$ we have

$\sum_{\mathfrak{F} \geq \mathfrak{F}_i} f(\mathfrak{F}) = 0$, the sum being ~~taken~~ over the q flags $\mathfrak{F} \in \mathfrak{M}(S)$ such that $\mathfrak{F} \geq \mathfrak{F}_i$.

<u>Proposition 3.47</u>

(i) There is an H-equivariant bijection $\theta : \mathcal{M}(J) \to \mathcal{A}(J)$ given by $\mathcal{F} \mapsto \mathcal{F} \cap A$.

(ii) There is an H-isomorphism

$\Theta : \{f : \mathcal{M}(S) \to \mathbb{Z}$ which satisfy $c_i^! \ (1 \leq i \leq n-1)\} \to St_H$ given by $\Theta(f)(\mathcal{F}) = f(\mathcal{F} \cap A)$.

<u>Corollary 3.48</u>

Θ^{-1} provides an injection $St_H \to St_G|_H$ of H-modules.

<u>Proof of §3.47</u>

(i) Let $\mathcal{A} \in \mathcal{A}(J)$. Suppose $\mathcal{B} = (A_0 \subset A_1 \subset \ldots \subset A_{n-2} \subset A)$ equals $\mathcal{A}$ if $1 \in J$ and belongs to the class $\mathcal{A}$ if $1 \notin J$. Let $(0 \subset W_1 \subset \ldots \subset W_{n-2} \subset W)$ be the flag through the origin parallel to $\mathcal{B}$, so $A_i = v_0 + W_i$ $(1 \leq i \leq n-2)$ where $A_0 = \{v_0\}$.

For $1 \leq i \leq n-2$ define $V_{i+1} = \langle v_0, W_i \rangle = \langle v_0 \rangle \oplus W_i$, and define $V_1 = \begin{cases} \langle v_0 \rangle & \text{if } 1 \in J \\ \langle \pi v_0 \rangle & \text{if } 1 \notin J \end{cases}$ (this is independent of the choice of A_0 in case $1 \notin J$). Then $\mathcal{F} = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$ has type J, meets A, and, since by §3.46(iii) $A \cap V_{i+1} = v_0 + \langle v_0, W_i \rangle \cap W = v_0 + W_i = A_i$, satisfies $\mathcal{F} \cap A = \mathcal{A}$. So θ is surjective.

Now suppose given $\mathcal{F} = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$ and $\mathcal{F}' = (0 \subset V_1' \subset \ldots \subset V_{n-1}' \subset V) \in \mathcal{M}(J)$ with $\theta(\mathcal{F}) = \theta(\mathcal{F}')$. Then the single points $\begin{cases} A \cap V_1,\ A \cap V_1' & (1 \in J) \\ \pi A \cap V_1,\ \pi A \cap V_1' & (1 \notin J) \end{cases}$ (see §3.46(i)) are the same, and since this point actually generates V_1 (and V_1') we see $V_1 = V_1'$. Also $A \cap V_{i+1} = A \cap V_{i+1}'$ $(1 \leq i \leq n-2)$ so, in the notation of §3.46, we may choose v_0 (for $\mathcal{F}$) and v_0' (for $\mathcal{F}'$) with $v_0 = v_0'$ (since $V_2 \cap A = V_2' \cap A$); and we have, by §3.46, $V_{i+1} \cap W = V_{i+1}' \cap W$ $(1 \leq i \leq n-2)$. Thus $V_{i+1} = \langle v_0 \rangle \oplus V_{i+1} \cap W = \langle v_0' \rangle \oplus V_{i+1}' \cap W = V_{i+1}'$. So $\mathcal{F} = \mathcal{F}'$, and θ is injective.

Now H acts on $\mathcal{M}(J)$ by restriction of the G-action : to see that H stabilises $\mathcal{M}(J)$ note that if $\mathfrak{F}$ meets A then there exists a unique $v \in \begin{cases} A \cap V_1 & (1 \in J) \\ \pi A \cap V_1 & (1 \notin J) \end{cases}$ and v generates V_1, so if $h \in H$ then $hv \in \begin{cases} h(A \cap V_1) = hA \cap hV_1 = A \cap hV_1 & (1 \in J) \\ h(\pi A \cap V_1) = h.\pi A \cap hV_1 = \pi A \cap hV_1 & (1 \notin J) \end{cases}$

so in fact $hV_1 = \langle hv \rangle$, and $h\mathfrak{F}$ meets A.

These facts, together with $h(A \cap V_{i+1}) = hA \cap hV_{i+1} = A \cap hV_{i+1}$ $(1 \leq i \leq n-2)$ show that $h.\theta(\mathfrak{F}) = \theta(h.\mathfrak{F})$ $(h \in H, \mathfrak{F} \in \mathcal{M}(J))$ and so θ is H-equivariant.

(ii) Using part(i) it remains only to check that for $1 \leq i \leq n-1$ a function $f : \mathcal{M}(S) \to \mathbb{Z}$ satisfies c_i' iff $\Theta(f)$ satisfies γ_i (using §3.44(ii)). For this we must show that if $\mathfrak{F}_i \in \mathcal{M}(\{i\}')$ then the q flags $\mathcal{A}$ such that $\mathcal{A} \geq \theta(\mathfrak{F}_i)$ are precisely $\theta(\mathfrak{F})$ as $\mathfrak{F}$ runs through the q flags $\mathfrak{F} \in \mathcal{M}(S)$ with $\mathfrak{F} \geq \mathfrak{F}_i$.

So suppose $\mathfrak{F}_i = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$. If $i \neq 1$ then the result is clear when we regard $\theta(\mathfrak{F}_i)$ as a pair $(v, \mathcal{G}_i)$ where $\mathcal{G}_i = (0 \subset W_1 \subset \ldots \subset W_{n-2} \subset W)$ with $W_i = V_{i+1} \cap W$ $(1 \leq i \leq n-2)$. If $i=1$ then the q flags $\mathfrak{F}$ are $(0 \subset W_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V)$, where W_1 runs through the q modules of type (h) with $V_1 \subset W_1 \subset V_2$. But $\theta(\mathfrak{F}) = (\{v_0\} \subset A_1 \subset \ldots \subset A_{n-2} \subset A)$ with $A_i = v_0 + V_i \cap W$, and v_0 the unique element of $A \cap W_1$, which in fact generates W_1; but $V_1 = W_1 = \langle \pi v_0 \rangle$ and $A \cap V_1 = \{\pi v_0\}$, so the flags $\theta(\mathfrak{F})$ are precisely the q flags in the class $\theta(\mathfrak{F}_i)$, i.e. they are just the flags $\mathcal{A}$. ∎

For the remainder of this chapter we shall consider only complex representations, and for convenience we shall write St_H instead of $St_H \otimes_{\mathbb{Z}} \mathbb{C}$, etc.

We can now give the principal results of this section.

Theorem 3.49

Let χ be a non-singular character of U. Then

(i) $\chi_U^H \cong St_H$; in particular, St_H is irreducible.

(ii) $< St_H , St_G|_H >_H = 1$.

(iii) $< \chi_U^G , St_G >_G = 1$.

Proof:

(i) Let $\mathcal{F}_0 = (0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$ where for $1 \leq j \leq n-1$, $U_j = < e_n, \ldots, e_{n-j+1} >$. Then $A \cap U_1 = \{e_n\}$, so $\mathcal{F}_0 \in M(S)$.

For each $g \in G$ let $f_g : \mathcal{F}(h^{n-1}) \to \mathbb{Z}$ denote the characteristic function of $g.\mathcal{F}_0$: $f_g(\mathcal{F}) = \begin{cases} 1 & \text{if } \mathcal{F} = g.\mathcal{F}_0 \\ 0 & \text{if } \mathcal{F} \neq g.\mathcal{F}_0 \end{cases}$ $(\mathcal{F} \in \mathcal{F}(h^{n-1}))$.

Then for any $h \in G$ and $\mathcal{F} \in \mathcal{F}(h^{n-1})$ we have $(h.f_g)(\mathcal{F}) = f_g(h^{-1}\mathcal{F}) = \begin{cases} 1 & \text{if } h^{-1}\mathcal{F} = g.\mathcal{F}_0 \\ 0 & \text{if } h^{-1}\mathcal{F} \neq g.\mathcal{F}_0 \end{cases}$, so $h.f_g = f_{hg}$.

Now $U \subset H$ so U stabilises $M(S)$, so if $u \in U$ then f_u is zero outside $M(S)$. For any $\chi \in \hat{U}$ define $F_\chi = \sum_{u \in U} \chi(u^{-1}) f_u$: this vanishes outside $M(S)$, so regard it as a function $F_\chi : M(S) \to \mathbb{C}$. It has the following properties.

Lemma 3.410

(i) If $u \in U$ then $u.F_\chi = \chi(u) F_\chi$.

(ii) F_χ satisfies c_i' $(1 \leq i \leq n-1)$ iff χ is non-singular.

Proof:

$$\begin{aligned} \text{(i)} \quad u.F_\chi &= \sum_{v \in U} \chi(v^{-1}) u.f_v \\ &= \sum_{v \in U} \chi((uv)^{-1}) \chi(u) f_{uv} \\ &= \chi(u) \sum_{w \in U} \chi(w^{-1}) f_w \quad \text{since the group U equals } \{w = uv : v \in U\} \text{ also} \\ &= \chi(u) F_\chi . \end{aligned}$$

(ii) Let $i \in S$, suppose given $\mathcal{F}_i \in M(\{i\}')$ and consider the flags ~~$\mathcal{F} \in M$~~(S) with $\mathcal{F} \geq \mathcal{F}_i$; we shall assume that at least one of these, say $\mathcal{F}_1$, has the form $\mathcal{F}_1 = u_1 . \mathcal{F}_0$ for some $u_1 \in U$, since otherwise $\sum_{\mathcal{F} \geq \mathcal{F}_i} F_\chi(\mathcal{F})$ is trivially zero.

Now the unique $\mathcal{F}_{0i} \in M(\{i\}')$ with $\mathcal{F}_0 \geq \mathcal{F}_{0i}$ is $(0 \subset U_1 \subset \ldots \subset W_{i+1} \subset \ldots \subset U_{n-1} \subset V)$ where $W_{i+1} = \langle e_n, \ldots, e_{n-i+1}, \pi e_{n-i} \rangle$, and the q flags $\mathcal{F}_r \in M(S)$ with $\mathcal{F}_r \geq \mathcal{F}_{0i}$ are $(0 \subset U_1 \subset \ldots \subset W_r \subset \ldots \subset U_{n-1} \subset V)$ $(r \in \bar{R})$, where $W_r = \langle e_n, \ldots, e_{n-i+1}, e_{n-i} + \pi^{h-1}\tilde{r}e_{n-i-1} \rangle$; so $\mathcal{F}_r = x_\alpha(\pi^{h-1}\tilde{r}).\mathcal{F}_0$ where $\alpha = \alpha_{n-i-1,n-i}$. Then the argument in part (ii) of the proof of §3.25 shows that $\mathcal{F}_i = u_1.\mathcal{F}_{0i}$ and the q flags $\mathcal{F}$ are $u_1 u.\mathcal{F}_0$ $(u \in U_\alpha(\pi^{h-1}R) = C_i$, say). Hence

$$\begin{aligned}\sum_{\mathcal{F} \geq \mathcal{F}_i} F_\chi(\mathcal{F}) &= \sum_{u \in C_i} F_\chi(u_1 u.\mathcal{F}_0) \\ &= \sum_{u \in C_i} \chi(u^{-1})\chi(u_1^{-1}).F_\chi(\mathcal{F}_0) \\ &= \chi(u_1^{-1}).\sum_{u \in C_i} \chi(u)\end{aligned}$$

which is zero iff $\chi|C_i \neq 1$ (regarding χ as a character of the abelian group U^{ab}).

Thus F_χ satisfies all the cycle conditions c_i' $(1 \leq i \leq n-1)$ iff $\chi|C_i \neq 1$ $(1 \leq i \leq n-1)$ iff χ is non-singular. ∎

Using the description of St_H given by §3.47(ii), the lemma shows that if χ is non-singular then F_χ is an eigenvector in St_H with eigenvalue given by χ, i.e. $\langle St_H|_U, \chi \rangle_U \geq 1$. So by Frobenius reciprocity $\langle St_H, \chi_U^H \rangle_H \geq 1$. ---(*).

Now $|G:H| = |\{\text{affine hyperplanes in A in general position}\}|$

$$= |\{(\underline{H}, v) : \underline{H} \text{ a hyperplane in } V, v \text{ a primitive vector on a line } L_{\underline{H}} \text{ with } \underline{H} \oplus L_{\underline{H}} = V\}|$$

(to each $\underline{H}$ we associate a complement $L_{\underline{H}}$; then v measures the distance of A from the origin, along $L_{\underline{H}}$).

So $\dim(\chi_U^H) = |H:U| = |G|/|U|.|G:H|$

$$= \left[\prod_{i=1}^{n} q^{(i-1)h}(q^{ih}-q^{i(h-1)})\right] \Big/ \left[\left(\prod_{i=1}^{n-1} q^{ih}\right).\frac{q^n-1}{q-1}q^{(n-1)(h-1)}.q^{h-1}(q-1)\right]$$

$$= \prod_{i=1}^{n-1}(q^{ih}-q^{i(h-1)}) \quad = \quad \dim(St_H),$$

using §1.4, §3.12, and §3.45.

Thus χ_U^H is irreducible (§3.33), is contained in St_H by (*), and has the same dimension as St_H; hence $\chi_U^H \cong St_H$.

(ii) Put $P(q) = \prod_{i=1}^{n-1}(q^i-1)q^{i(h-1)-1}$. Then $\dim(St_G) = P(q).\frac{q^n-1}{q-1}$ by §3.24; and by §3.45 $\dim(St_H) = P(q).q^{n-1}$.

Hence
$$\begin{aligned}\dim(St_G)/\dim(St_H) &= (q^n-1)/(q-1)q^{n-1}\\ &= (q^{n-1}+q^{n-2}+ \dots +q+1)/q^{n-1}\\ &= 1+q^{-1}+q^{-2}+ \dots +q^{-(n-1)}\\ &< \sum_{r=0}^{\infty} q^{-r}\\ &= 1/(1-q^{-1})\\ &= 1 + 1/(q-1)\\ &< 2 \quad \text{for all } q \geq 2,\ n \geq 1.\end{aligned}$$

But St_H is irreducible by (i) and $< St_H\ ,\ St_G|_H >_H \ \geq 1$ by §3.48, so in fact $< St_H\ ,\ St_G|_H >_H \ = 1$.

(iii) By (i) and (ii) $< \chi_U^H\ ,\ St_G|_H >_H \ = 1$, so by Frobenius reciprocity $< \chi_U^G\ ,\ St_G >_G \ = 1$. ∎

Remarks

(i) The construction of Steinberg representations St_H, St_G in case R is a field (h=1) is given in [13], where there is also a direct proof of the irreducibility of St_H, using cycle conditions; such a proof may also exist in our case $(h \geq 2)$, (cf.§3.25), but is likely to be more difficult than the method of establishing an isomorphism $St_H \cong \chi_U^H$ for any non-singular character χ of U. The latter result also holds if h=1.

(ii) It is also true for all $h \geq 1$ that $\dim(St_H)$ equals the dimension of an unramified discrete series representation D of G. Considering the results of [8] in case h=1 we may ask whether $D|_H \cong St_H$ for any such D. This would be the case if the following results hold (as they do in case h=1) :

(1) an irreducible representation E of G is in the unramified discrete series of G iff the only linear characters of U contained in the restriction of E to U are the non-singular ones.

(2) if χ is a non-singular character of U then χ_U^G is multiplicity-free.

In case n=2 these results do hold : note that the restriction to U of a ramified discrete series representation of G contains both singular and non-singular charatters of U (this follows from the character tables in [16] if $p \neq 2$).

We shall show that (2) also holds in case n=3 (see §3.5).

(iii) It is true in case h=1 that if χ is a non-singular character of U then St_G is the unique common irreducible component of χ_U^G and 1_B^G. Using §3.49(iii) we can show that this is also true for general $h \geq 1$ if (2) holds (see §3.6); incidentally, §3.49(iii) provides a proof of the irreducibility of St_G which is independent of that of §3.25/6.

The foregoing remarks provide some justification for regarding our representations St_H, St_G as analogues for $h \geq 2$ of the usual Steinberg representations of the case h=1.

3.5 The 'Gelfand-Graev representation'

The purpose of this section is to discuss the analogues, for $h \geq 2$, of the results of Gelfand and Graev ([7]) concerning the representations χ_U^G. First, we point out that their method of proof may be carried through to show that, given an irreducible representation E of G, there exists a character χ of U for which χ_U^G contains E. More difficult to carry through is their second result, which we have only been able to prove in special cases :

Theorem 3.51

Let χ be a non-singular character of U; assume n=2 or 3. Then χ_U^G is multiplicity-free.

Proof:

As usual, using Frobenius reciprocity, we may identify $\mathrm{End}_G(\chi_U^G)$ with $\mathcal{E} = \{f:G \to \mathbb{C}$ such that $f(ux) = \chi(u)f(x) = f(xu)$ for $x\in G, u\in U\}$. We must show that $\mathcal{E}$ is commutative.

Let $\theta : G \to G$ be the antiautomorphism which reflects the matrix $g\in G$ in its secondary diagonal; so θ consists of transposing the matrix g and conjugating it by w_0 (in either order). Since χ_U^G is independent of the particular choice of non-singular χ we may assume that χ is given by $\chi(\prod_{\alpha>0} x_\alpha(r_\alpha)) = \psi(\sum_{\alpha\in\Sigma} r_\alpha)$ for some fixed non-singular character ψ of K.

Then $\theta(U) = U$ and $\chi\circ\theta = \chi$ on U -----(*)

We shall produce a set X of U-U double coset representatives in G such that if $x\in X$ and $f\in\mathcal{E}$ then

$$\text{either} \quad f(x) = 0 \quad \text{or} \quad \theta(x) = x \qquad \text{-----(**)}.$$

Assuming this done we may prove the theorem as follows. The antiautomorphism θ acts on $\mathcal{E}$ by $(\theta.f)(g) = f(\theta(g))$ $(g\in G, f\in\mathcal{E})$ but is also the identity, since the values of $f\in\mathcal{E}$ are determined by its values on $\{ux : u\in U, x\in X\}$ and on here we have $(\theta.f)(ux) = f(\theta(ux)) = f(\theta(x)\theta(u)) = f(x.\theta(u)) = f(x).\chi(\theta(u)) = f(x)\chi(u) = f(ux)$. Hence for $f_1, f_2\in\mathcal{E}$ we have $f_1\circ f_2 = \theta(f_1\circ f_2) = \theta(f_2)\circ\theta(f_1) = f_2\circ f_1$, so $\mathcal{E}$ is commutative.

It remains to prove (**).

Lemma 3.52

The set $X_1 = \{vwt : t\in T, w\in W, v \in U^-(\pi K) \cap {}^wU^-(\pi K)\}$ forms a complete set of representatives for distinct U-U cosets in G.

Proof:

By §1.46 and the fact that $B = T.U$ (§1.27) the set $\{uvwt : t\in T,\ w\in W,\ u\in U\cap {}^{w}U^{-},\ v\in U^{-}(\pi R)\cap {}^{w}U^{-}(\pi R)\}$ forms a complete set of representatives for distinct left cosets of G by U. So certainly X_1 contains representatives for all U-U cosets in G. It remains to show that if $u_1 \in U\cap {}^{w}U^{-}$, $u_2 \in U$, $x_1, x_2 \in X_1$ and $u_1x_1u_2 = x_2$ then $x_1 = x_2$.

So suppose $x_1 = v_1w_1t_1$, $x_2 = v_2w_2t_2$. Then $\overline{u_1x_1u_2} = \bar{u}_1w_1\bar{t}_1\bar{u}_2$ lies in the $\bar{B}$-$\bar{B}$ coset of $\bar{G}$ given by w_1 (§1.47) and $\bar{x}_2$ lies in that given by w_2. So $w_1 = w_2 = w$, say. Hence

$$\begin{aligned} v_2^{-1}u_1v_1 &= wt_2u_2^{-1}t_1^{-1}w^{-1} \\ &= w(t_2t_1^{-1})w^{-1}.w(t_1u_2^{-1}t_1^{-1})w^{-1} \\ &\in {}^{w}T.{}^{w}U = T.{}^{w}U, \end{aligned}$$

so $v_2^{-1}u_1v_1 \in {}^{w}U^{-} \cap T.{}^{w}U = 1$.

Thus $w(t_2t_1^{-1})w^{-1} = 1 = w(t_1u_2^{-1}t_1^{-1})w^{-1}$, whence $t_1 = t_2$ and $u_2 = 1$; and $u_1 = v_2v_1^{-1} \in (U\cap {}^{w}U^{-}) \cap (U^{-}\cap {}^{w}U^{-}) = 1$, so $v_1 = v_2$ and $u_1 = 1$. Hence $x_1 = x_2$. ∎

Lemma 3.53

Let $x = vwt \in X_1$, $f \in \mathcal{E}$, and suppose $f(x) \neq 0$. Then there exists $J \subset S$ such that $w = w_0w_J$.

Proof:

Let $u = x_\alpha(r)$ with $r \| \pi^{h-1}$, for some $\alpha \in \Phi$. Then $xux^{-1} = vw.tx_\alpha(r)t^{-1}.w^{-1}v^{-1} = v.wx_\alpha(\alpha(t).r)w^{-1}.v^{-1} = vx_{w\alpha}(\alpha(t).r)v^{-1} = x_{w\alpha}(\alpha(t).r)$, using §1.56, §1.36, and §1.34.

Now if $\alpha > 0$ $\chi(u)f(x) = f(xu) = f(xux^{-1}.x) = \chi(xux^{-1})f(x)$, so if $f(x) \neq 0$ and $w\alpha > 0$ we have $\chi(u) = \chi(xux^{-1})$. But if $\alpha \in \Sigma$ then since χ is non-singular, $\chi(u) = \psi(r) \neq 1$, so $\chi(xux^{-1}) \neq 1$, and so $w\alpha \in \Sigma$ since $\alpha(t).r \| \pi^{h-1}$.

It now follows from Lemma 89 (p.257) of [18] that $w = w_0w_J$ for some $J \subset S$. ∎

<u>Lemma 3.54</u>

(i) If $t \in T$ then $\theta(t) = w_0 t w_0$.

(ii) If $w \in W$ then $\theta(w) = w_0 w^{-1} w_0$

$= w$ in case $w = w_0 w_J$ for some $J \subset S$.

(iii) Suppose that t and J are as in §3.53. Then

$t w_{\{i\}} = w_{\{i\}} t$ for all $i \in J$, and hence $t w_J = w_J t$.

<u>Proof</u>:

(i) The transpose matrix of t is t so (i) is immediate.

(ii) w is a permutation matrix : an entry 1 in the (i,j) place (i'th row, j'th column) means that $w(j) = i$. Transposing w brings the (i,j) entry to the (j,i) place. So the transpose of w is w^{-1}.

(iii) If $\alpha \in \Sigma$ and $w\alpha \in \Sigma$ then, as in §3.53, $w = w_0 w_J$ and so $\alpha = \alpha_{i,i+1}$ for some $i \in J$; moreover $\alpha(t) = 1$, since $\psi(r) = \psi(\alpha(t).r)$. Thus $t_i = t_{i+1}$ for $i \in J$ ($\alpha(t) = t_i t_{i+1}^{-1}$ as in §1.5). Hence $t_j = t_{w_{\{i\}}(j)}$ for all j, and $i \in J$, and so $t = w_{\{i\}} t w_{\{i\}}^{-1}$ for all $i \in J$. ∎

<u>Corollary 3.55</u>

Suppose $x = vwt \in X_1$, $f \in \Sigma$, and $f(x) \neq 0$.

Assume that $\theta(v) = t^{-1} w^{-1} vwt$.

Then $\theta(x) = x$.

<u>Proof</u>:

$$\begin{aligned}
\theta(x) = \theta(t)\theta(w)\theta(v) &= w_0 t w_0 . w . \theta(v) \quad \text{by §3.53, §3.54(i)/(ii)} \\
&= w_0 t w_J \theta(v) \\
&= w_0 w_J t.\ t^{-1} w^{-1} vwt \quad \text{by §3.54(iii)} \\
&= wt . t^{-1} w^{-1} . vwt \\
&= x.
\end{aligned}$$
∎

Unfortunately, the assumption of §3.55 does not always hold;

and since $\theta(x) = x$ iff the assumption holds, we must replace v by an element $v' \in U^-(\pi R)$ for which the assumption does hold, and for which there exist $u_1, u_2 \in U$ such that $x' = v'wt = u_1 vwtu_2$. Then $f(x) \neq 0$ iff $f(x') \neq 0$, so §3.55 shows that $\theta(x') = x'$, and thus (**) will be proved and hence also Theorem 3.51.

We proceed to consider the cases n=2,3 showing explicitly that for each $x = vwt \in X_1$ for which $f(x) \neq 0$ either $\theta(x)=x$ or else there exists $v' \in U^-(\pi R)$ such that, putting $x'=v'wt$, we have $\theta(x') = x'$ and $x' = u_1 x u_2$ for some $u_1, u_2 \in U$.

By §3.53 we need only consider w of the form $w_0 w_J$ $(J \subset S)$.

<u>Case n=2</u>

(i) $w = w_0 w_\emptyset = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$x = vwt = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \quad \text{for some}$$

$a, b \in R^*$. Hence $\theta(x) = x$.

(ii) $w = w_0 w_{\{1\}} = 1$.

$$x = vwt = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (r \in \pi R; a, b \in R^*).$$

By §3.54(iii), if $f(x) \neq 0$ then

$$\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

so $a = b$; hence $x = \begin{pmatrix} a & 0 \\ ar & a \end{pmatrix}$ and so $\theta(x) = x$.

<u>Case n=3</u>

(i) $w = w_0 w_\emptyset = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then $w\Phi^- \cap \Phi^- = \emptyset$. Hence

$$x = vwt = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}, \text{ for}$$

some $a, b, c \in R^*$. So $\theta(x) = x$.

(ii) $w = w_0 w_{\{1\}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then $w\Phi^- \cap \Phi^- = \{\alpha_{32}\}$ so $v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{pmatrix}$ for some $r \in \pi R$.

But by §3.54(iii), if $f(x) \neq 0$ then

$$\begin{pmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & c \end{pmatrix},$$

so $a = b$; hence $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ ar & a & 0 \end{pmatrix}$,

and so $\theta(x) = x$.

(iii) $w = w_0 w_{\{2\}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Then $w\Phi^- \cap \Phi^- = \{\alpha_{21}\}$ so $v = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for some $r \in \pi R$.

But by §3.54(iii), if $f(x) \neq 0$ then

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & c \\ 0 & b & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

so $b = c$; hence $x = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ 0 & br & b \\ a & 0 & 0 \end{pmatrix}$,

and so $\theta(x) = x$.

(iv) $w = w_0 w_{\{1,2\}} = 1$.

By §3.54(iii), if $f(x) \neq 0$ then $tw_{\{i\}} = w_{\{i\}}t$ for $i=1,2$ so by the calculations in (ii) and (iii) above, with $t = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,

we have $a = b = c$, and hence $x = \begin{pmatrix} a & 0 & 0 \\ k & a & 0 \\ l & m & a \end{pmatrix}$ for some $k,l,m \in \pi R$.

If $h=1$ then $k=l=m=0$ and $\theta(x) = x$, so assume $h \geq 2$. Also if $k=m$ then $\theta(x) = x$; so suppose $k \neq m$.

Suppose also $l \| \pi^\lambda$ and $(k-m) \| \pi^\nu$ (then $\nu < h$).

<u>Case 1</u> : $\lambda > \nu$.

Put $u = \pi^{h-\lambda}$, $y = kua^{-1}$, and $z = mua^{-1}$; then $lu = 0$ but $y \neq z$.

Now $\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ k & a & 0 \\ l & m & a \end{pmatrix} = \begin{pmatrix} a & um & au \\ k & a+my & ay \\ l & m & a \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ k & a & 0 \\ l & m & a \end{pmatrix}\begin{pmatrix} 1 & z & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an

identity of the form $u_1 x = x u_2$ $(u_1, u_2 \in U)$, so if $f(x) \neq 0$

then we have $\chi(u_1) = \chi(u_2)$, i.e. $\psi(y) = \psi(z)$, and so since ψ is non-singular we have $y = z$, a contradiction. So $f(x)=0$ for all $f \in \Xi$.

Case 2 : $\nu \geq \lambda$ and $l \nmid k$ or $l \nmid m$ (or both).

Then since $\nu \geq \lambda$, $k \| m \| \pi^{\mu}$, say, with $\mu < \lambda$.

Write $\Delta = al - km$ and suppose $\Delta \| \pi^{\sigma}$; then $\sigma > \mu$.

We remark here that, given $r, s \in R$ then either $r|s$ or $s|r$ (or both, in which case $r \| s$); in fact if $r = r_0\pi^{\alpha}$, $s = s_0\pi^{\beta}$ ($r_0, s_0 \in R^*$) then $r|s$ iff $\beta \geq \alpha$, and we shall conventionally write $r^{-1}s$ to mean $r_0^{-1}s_0\pi^{\beta-\alpha}$.

Then $lk^{-1}, lm^{-1}, \Delta k^{-1}, \Delta m^{-1}$ all make sense.

(a) First consider the case when $\lambda + \sigma > \nu + \mu$.

Put $u = \pi^{h+\mu-\lambda-\sigma}$. Then $u \neq 0$ and $\Delta ulk^{-1} = 0 = \Delta ulm^{-1}$.

Then
$$\begin{pmatrix} 1 & -ulk^{-1} & u \\ 0 & 1 & -ua^{-1}\Delta m^{-1} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ k & a & 0 \\ l & m & a \end{pmatrix} = \begin{pmatrix} a & -u\Delta k^{-1} & au \\ k & a-ua^{-1}\Delta & -u\Delta m^{-1} \\ l & m & a \end{pmatrix}$$
$$= \begin{pmatrix} a & 0 & 0 \\ k & a & 0 \\ l & m & a \end{pmatrix}\begin{pmatrix} 1 & -ua^{-1}\Delta k^{-1} & u \\ 0 & 1 & -ulm^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

is an identity of the form $u_1x = xu_2$ ($u_1, u_2 \in U$), so if $f(x) \neq 0$ then we have $\chi(u_1) = \chi(u_2)$; so since ψ is non-singular we have $(-ulk^{-1} - ua^{-1}\Delta m^{-1}) = (-ua^{-1}\Delta k^{-1} - ulm^{-1})$. Hence $uk^{-1}(a^{-1}\Delta - l) = um^{-1}(a^{-1}\Delta - l)$, i.e. $-ua^{-1}m = -ua^{-1}k$, and so $(k-m)\pi^{h+\mu-\lambda-\sigma} = 0$. So $h+\mu-\lambda-\sigma \geq h-\nu$. But this contradicts $\lambda+\sigma > \nu+\mu$. So $f(x) = 0$ for all $f \in \Xi$.

(b) Now consider the case when $\lambda + \sigma \leq \nu + \mu$.

We may write $k = f + r\pi^{\nu}$ and $m = f + s\pi^{\nu}$ with $f \| \pi^{\mu}$ and $(r-s) \in R^*$.

Put $x = a(m-f)\Delta^{-1}$, $w = a(k-f)\Delta^{-1}$, $y = (f-k)l^{-1}$, $v = (f-m)l^{-1}$, and $z = a(k-m)f\Delta^{-1}l^{-1}$; note that these make sense since $\sigma \leq \nu + (\mu - \lambda) < \nu$, and $\nu \geq \lambda$. Then

$$\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}\begin{pmatrix}a&0&0\\k&a&0\\l&m&a\end{pmatrix}\begin{pmatrix}1&v&0\\0&1&w\\0&0&1\end{pmatrix}=\begin{pmatrix}a+kx+lz&ax+mz&az\\k+ly&a+my&ay\\l&m&a\end{pmatrix}\begin{pmatrix}1&v&0\\0&1&w\\0&0&1\end{pmatrix}$$

$= x'$, say, and the entries x'_{ij} of x' satisfy the following :

(1) $x'_{11}-x'_{33} = (a+kx+lz) - (a+mw)$

$= ka(m-f)\Delta^{-1} + la(k-m)f\Delta^{-1}l^{-1} - ma(k-f)\Delta^{-1}$

$= 0.$

(2) $x'_{12}-x'_{23} = v(a+kx+lz) + ax + mz - w(a+my) - ay$

$= a(f-m)l^{-1} + k(f-m)l^{-1}a(m-f)\Delta^{-1} + l(f-m)l^{-1}a(k-m)f\Delta^{-1}l^{-1}$

$+ a^2(m-f)\Delta^{-1} + ma(k-m)f\Delta^{-1}l^{-1} - a^2(k-f)\Delta^{-1}$

$- ma(k-f)\Delta^{-1}(f-k)l^{-1} - a(f-k)l^{-1}$

$= a\Delta^{-1}l^{-1}\Big[(al-km)(k-m) + a(m-k)l - k(f-m)^2$

$+ f(f-m)(k-m) + fm(k-m) + m(k-f)^2\Big]$

$= 0.$

(3) $x'_{21}-x'_{32} = (k+ly) - (lv+m)$

$= k + l(f-k)l^{-1} - l(f-m)l^{-1} - m$

$= 0.$

So $\theta(x') = x'$, and $x' = u_1xu_2$ with $u_1,u_2 \in U$.

<u>Case 3</u> : $l|k$ and $l|m$.

Put $k = cl$, $m = dl$ and write $y = \dfrac{a(d-c)}{a-cdl}$.

Then $\begin{pmatrix}1&y&-cy\\0&1&-c\\0&0&1\end{pmatrix}\begin{pmatrix}a&0&0\\k&a&0\\l&m&a\end{pmatrix}\begin{pmatrix}1&-d&0\\0&1&0\\0&0&1\end{pmatrix}$

$$=\begin{pmatrix}a&ay-mcy&-acy\\0&a-mc&-ac\\l&m&a\end{pmatrix}\begin{pmatrix}1&-d&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}a&-ac&-acy\\0&a-mc&-ac\\l&0&a\end{pmatrix}=x', \text{ say.}$$

Then $\theta(x') = x'$, and $x' = u_1xu_2$ with $u_1,u_2 \in U$.

Theorem 3.51 is now proved. ■

For any non-singular character χ of U let i be the number of irreducible components of χ_U^G, and let $r = q^{nh-1}(q-1)$, which is the number of regular conjugacy classes of G (§1.57). It is known ([18],p.263) that if h=1 then i=r.

Proposition 3.56

If $n = 2$ then $i = r$.

If $n = 3$ then $i \geq r$, with equality iff $h = 1$.

Proof:

$i = |\{x \in X_1 : f(x) \neq 0 \text{ for some } f \in \xi\}|$. To show that, given $x \in X_1$, $f(x) \neq 0$ for some $f \in \xi$ we must show that any relation $u_1 x = x u_2$ $(u_1, u_2 \in U)$ implies $\chi(u_1) = \chi(u_2)$.

Case n=2

(i) If $\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & b\\ a & 0\end{pmatrix} = \begin{pmatrix}0 & b\\ a & 0\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$, i.e. $\begin{pmatrix}ua & b\\ a & 0\end{pmatrix} = \begin{pmatrix}0 & b\\ a & av\end{pmatrix}$,

then $ua = av = 0$, so $u = v = 0$, and so $\chi\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \chi\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$.

(ii) If $\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ r & a\end{pmatrix} = \begin{pmatrix}a & 0\\ r & a\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$, i.e. $\begin{pmatrix}a+ru & ua\\ r & a\end{pmatrix} =$ $\begin{pmatrix}a & av\\ r & a+rv\end{pmatrix}$, then $ua = av$, so $u = v$ and $\chi\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \chi\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$.

So the set $\left\{\begin{pmatrix}0 & b\\ a & 0\end{pmatrix}, \begin{pmatrix}a & 0\\ r & a\end{pmatrix} : a, b \in R^*, r \in R\right\}$ is a basis for ξ, and has $q^{2(h-1)}(q-1)^2 + q^{h-1}.q^{h-1}(q-1) = q^{2h-1}(q-1)$ elements. So the dimension i of ξ equals r.

Case n=3

(i) If $\begin{pmatrix}1 & u & v\\ 0 & 1 & w\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & c\\ 0 & b & 0\\ a & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & c\\ 0 & b & 0\\ a & 0 & 0\end{pmatrix}\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$, then

$\begin{pmatrix}va & ub & c\\ wa & b & 0\\ a & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & c\\ 0 & b & bz\\ a & ax & ay\end{pmatrix}$, so $x = y = z = u = v = w = 0$.

Hence $\chi\begin{pmatrix}1 & u & v\\ 0 & 1 & w\\ 0 & 0 & 1\end{pmatrix} = \chi\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$.

(ii) If $\begin{pmatrix}1 & u & v\\ 0 & 1 & w\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & c\\ a & 0 & 0\\ r & a & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & c\\ a & 0 & 0\\ r & a & 0\end{pmatrix}\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$, then

$\begin{pmatrix}ua+vr & va & c\\ a+wr & wa & 0\\ r & a & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & c\\ a & ax & ay\\ r & rx+a & ry+az\end{pmatrix}$, so $av = ay = 0$ (so $v = y = 0$)

$ua+vr = 0 = ry+az$ (so $u = 0 = z$) and $wa = ax$, so $w = x$.

Hence $\chi\begin{pmatrix}1 & u & v\\ 0 & 1 & w\\ 0 & 0 & 1\end{pmatrix} = \psi(w) = \psi(x) = \chi\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}$.

(iii) If $\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}\begin{pmatrix}0&b&0\\0&r&b\\a&0&0\end{pmatrix}=\begin{pmatrix}0&b&0\\0&r&b\\a&0&0\end{pmatrix}\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$, then

$\begin{pmatrix}av&b+ur&bu\\aw&r&b\\a&0&0\end{pmatrix}=\begin{pmatrix}0&b&bz\\0&r&rz+b\\a&ax&ay\end{pmatrix}$, so $av = ay = 0 = aw = ax$ and

$bu = bz$, so $u = z$ and $\chi\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}=\psi(u)=\psi(z)=\chi\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$.

(iv) If $\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}\begin{pmatrix}a&0&0\\k&a&0\\l&m&a\end{pmatrix}=\begin{pmatrix}a&0&0\\k&a&0\\l&m&a\end{pmatrix}\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$, then

$$\begin{cases}uk+vl = 0 = ly+mz\ ,\ va = ay,\ wl = lx = 0,\\ wm = kx,\ au+vm = ax,\ wa = ky+az.\end{cases}$$

Hence $\begin{cases}v = y,\quad x = u+mva^{-1},\ w = z+kva^{-1},\\ lw = 0 = lx,\ ku = -lv = mz.\end{cases}$

So $(u+w) - (x+z) = (u-x) + (w-z) = (k-m)va^{-1}$.

Thus if $k = m$ then $u+w = x+z$ and $\chi\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}=\chi\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$.

So the following is a subset of a basis of $\mathcal{E}$:

$$\left\{\begin{pmatrix}0&0&c\\0&b&0\\a&0&0\end{pmatrix},\begin{pmatrix}0&0&c\\a&0&0\\r&a&0\end{pmatrix},\begin{pmatrix}0&b&0\\0&r&b\\a&0&0\end{pmatrix},\begin{pmatrix}a&0&0\\k&a&0\\l&k&a\end{pmatrix} : \begin{matrix}a,b,c\in R^* \text{ and}\\ r,k,l\in\pi R.\end{matrix}\right\},$$

and this set has $q^{h-1}(q-1).q^{2(h-1)}\left[(q-1)^2+2(q-1)+1\right] =$ $q^{3h-1}(q-1) = r$ elements.

In case $h=1$ there are no other $x \in X_1$ with $f(x) \neq 0$ for some $f \in \mathcal{E}$ (since $\pi R=0$), so $i=r$ in this case. But if $h \geq 2$ then $\mathcal{E}$ contains more elements in a basis; for example, in case (iv), if $k\nmid m$ and $l|k, l|m$ (say $k=cl$, $m=dl$ as before) then writing $\Delta = al-km$ we have $\Delta \| l$ and $\Delta ua^{-1} = lu+lmva^{-1} = lx$ $=0$, whence $lu = 0$. Similarly $\Delta za^{-1} = lz+lkva^{-1} = lw = 0$, whence $lz = 0$. Hence $u+w = (u-cdlua^{-1}) + (z+clva^{-1}) =$ $(u+dlva^{-1}) + (z-cdlza^{-1}) = x+z$, so $\chi\begin{pmatrix}1&u&v\\0&1&w\\0&0&1\end{pmatrix}=\chi\begin{pmatrix}1&x&y\\0&1&z\\0&0&1\end{pmatrix}$.

So the elements $\begin{pmatrix}a&0&0\\cl&a&0\\l&dl&a\end{pmatrix}$ may also be included in a basis of $\mathcal{E}$, and so $i > r$. ∎

3.6 St_G as a 'principal series' representation

Let λ be a character of $T = GL(W_1) \times \dots \times GL(W_n)$ (§1.24). Then $\lambda = (\lambda_1,\dots,\lambda_n)$ where λ_i is a character of $GL(W_i) \cong R^*$. For $1 \leq i,j \leq n$ $(i \neq j)$ define $k_{ij}(\lambda)$ to be the smallest integer k such that $(\lambda_i - \lambda_j)|R^{(k)}$ is trivial. Write $k_{ij}(\lambda) = k_\alpha(\lambda)$ when $\alpha_{ij} = \alpha \in \Phi$ (cf.§1.5). We shall say that λ is in <u>general position</u> if $k_\alpha(\lambda) = h$ for all $\alpha \in \Phi$.

The following is essentially contained in [10]. First, we may re-order the λ_i (by the action of W) so that the sequence $\mathfrak{s}_\lambda = (k_{12}(\lambda),\dots,k_{n-1,n}(\lambda))$ determines $k_\alpha(\lambda)$ for all $\alpha \in \Phi$; in fact if $1 \leq i < j \leq n$ then $k_{ij}(\lambda) = \max\limits_{i \leq r < j} (k_{r,r+1}(\lambda))$. Then, in the partially ordered set of subgroups $H_\mathfrak{s}$ of G (see §2.1) $H_{\mathfrak{s}_\lambda}$ is the unique maximal one to which we can extend λ trivially. Note that if λ is in general position then $H_{\mathfrak{s}_\lambda} = B$.

Proposition 3.61

$\lambda^G_{H_{\mathfrak{s}_\lambda}}$ is irreducible.

This is proved by showing $\langle \lambda^G_B, \lambda^G_{H_{\mathfrak{s}_\lambda}} \rangle_G = 1$ (see [10], Thm.1). Note that if $\lambda = 1$ then $\lambda^G_B = 1^G_B$ and $\lambda^G_{H_{\mathfrak{s}_\lambda}} = 1$.

Now λ^G_B contains an interesting irreducible representation at the 'opposite end' of its spectrum (at least if n=2 or 3), which is just St_G in case $\lambda = 1$:

Proposition 3.62

Let χ be a non-singular character of U and assume that χ^G_U is multiplicity-free. Then $\langle \lambda^G_B, \chi^G_U \rangle_G = 1$.

Proof:

If λ, μ are characters of T then $\lambda^G_B|_U = 1 = \mu^G_B|_U$, so $\langle \lambda^G_B, \chi^G_U \rangle_G = \langle 1, \chi \rangle_U = \langle \mu^G_B, \chi^G_U \rangle_G$ by Frobenius reciprocity. Hence $\langle \lambda^G_B, \chi^G_U \rangle_G$ is independent of the choice of λ.

But $< 1_B^G, \chi_U^G >_G \;\geq\; < St_G, \chi_U^G >_G = 1$, by §3.49(iii); and if λ is in general position then λ_B^G is irreducible, by §3.61, and so $1 \geq <\lambda_B^G, \chi_U^G>_G$ since χ_U^G is multiplicity-free. Hence $1 = <\lambda_B^G, \chi_U^G>_G$ for all λ. ∎

Chapter 4

A geometric interpretation of S_G and St_G

In this chapter we express S_G and St_G as alternating sums of permutation representations of G on suitable collections of subcomplexes of a certain finite complex X(V). We construct X(V) in §4.1 as a subcomplex of an (affine) Bruhat-Tits building. With these geometric descriptions we are able to use the combinatorial results of §1.44 to show that S_G is a subrepresentation of 1_B^G and contains St_G (except, perhaps, if q=2).

In §4.4 we consider as examples the cases n=2 and 3, and in particular we use the geometry to describe the connection between St_G for $h \geq 2$ and the usual Steinberg representation for the case h=1. Finally, in §4.5 we show how such geometric definitions of S_G and St_G may be made for more general groups.

4.1 The Bruhat-Tits building (for GL_n).

As in §1.0 let K be a non-archimedean local field, $\mathcal{O}$ its maximal compact subring and ξ a generator of the maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Let V be a vector space of dimension n over K.

A lattice in V is a free $\mathcal{O}$-module of rank n, contained in V.

Lemma 4.11

Let L,L' be lattices in V. Then there exists a unique $m \in \mathbb{Z}$ such that $\begin{cases} \text{(i) } \xi^m L' \subset L \\ \text{(ii) } l(L/\xi^m L') \leq n-1. \end{cases}$

Proof:

We can choose a basis $\{e_1,\dots,e_n\}$ for V which also serves as a basis for L as a free $\mathcal{O}$-module. Then any $x \in L'$ has the form $x = \sum_{i=1}^{n} a_i e_i$ where $a_i \in K$ $(1 \leq i \leq n)$. But given $a \in K$ there exists $k \in \mathbb{Z}$ such that $\xi^k a \in \mathcal{O}$, so there exists $m \in \mathbb{Z}$ such that $\xi^m L' \subset L$ (cf.[19],Thm.2,p.30).

In the same way, given such m there exists $m' \in \mathbb{Z}$ ($m' \geq 0$ in fact) such that $\xi^{m'}L \subset \xi^m L'$; so $\xi^{m'}(L/\xi^m L') = 0$ and hence $L/\xi^m L'$ is a module over $\mathcal{O}/\mathfrak{p}^h$ for any $h \geq m'$. Suppose $L/\xi^m L'$ has type $(a_1,\ldots,a_n)$ where $h \geq a_1 \geq \ldots \geq a_n \geq 0$.

Let $p: L \to L/\xi^m L'$ be the natural projection. Then $L/\xi^m L' = \langle p(e_1),\ldots,p(e_n)\rangle$ and $p(e_i)$ has order ξ^{a_i} (after suitable re-ordering of the e_i). Moreover, the module $M = \langle \xi^{a_1-a_n}p(e_1),\ldots,\xi^{a_{n-1}-a_n}p(e_{n-1}),p(e_n)\rangle$ has type $(a_n,\ldots,a_n)$ and $L'' = p^{-1}(M) = \langle \xi^{a_1-a_n}.e_1,\ldots,\xi^{a_{n-1}-a_n}.e_{n-1},e_n\rangle$ is a lattice in V contained in L.

But $\ker(p) = \xi^m L' \subset L''$ so $L/L'' \sim (a_1-a_n,\ldots,a_{n-1}-a_n)$ and $l(L/L'') \leq n-1$. Also $\xi^{a_n}L'' \subset \xi^m L' \subset L''$ so the exact sequence $0 \to \xi^m L'/\xi^{a_n}L'' \to L''/\xi^{a_n}L'' \to L''/\xi^m L' \to 0$ (in which the third and fourth terms both have type $(a_n,\ldots,a_n)$) shows that $\xi^m L' = \xi^{a_n}L''$.

Hence $\xi^{m-a_n}L' = L'' \subset L$ and $l(L/L'') \leq n-1$. So replacing m by $m+a_n$ we see properties (i) and (ii) hold.

To prove uniqueness suppose m_1, m_2 both satisfy (i) and (ii). Assume $m_1 \geq m_2$ so that $\xi^{m_1}L' \subset \xi^{m_2}L'$ and consider the exact sequence $0 \to \xi^{m_2}L'/\xi^{m_1}L' \to L/\xi^{m_1}L' \to L/\xi^{m_2}L' \to 0$. The second term has type $(a,\ldots,a)$ (length n if $a \neq 0$) where $a = m_1 - m_2$., but it also injects into the module $L/\xi^{m_1}L'$ of length $\leq n-1$. So $a=0$, i.e. $m_1 = m_2$. ■

<u>Definition</u>

Lattices L, L' in V are equivalent iff there exists $a \in K^*$ such that $L = aL'$ (iff there exists $m \in \mathbb{Z}$ such that $L = \xi^m L'$).

(Clearly this is an equivalence relation).

We make this definition because of the following :

Lemma 4.12

The isomorphism type of the module $L/\xi^m L'$ of §4.11 depends only on the equivalence classes of L and L'.

Proof:

Consider the lattices $\xi^r L, \xi^{r'} L'$ where $r, r' \in \mathbb{Z}$. By §4.11 there exists a unique $m' \in \mathbb{Z}$ such that (i)': $\xi^{m'+r'} L' \subset \xi^r L$ and (ii)': $l(\xi^r L/\xi^{m'+r'} L') \leq n-1$.

But (i)' holds iff $\xi^{m'+r'-r} L' \subset L$ and (ii)' holds iff $l(L/\xi^{m'+r'-r} L') \leq n-1$, since $\xi^r L/\xi^{m'+r'} L' \cong L/\xi^{m'+r'-r} L'$. So by §4.11 $m'+r'-r = m$; hence $L/\xi^m L' \cong \xi^r L/\xi^{m'}.\xi^{r'} L'$. ■

Now recall that the height $h = ht(M)$ of a finite $\mathcal{O}$-module M is the positive or zero integer defined by $\xi^h M=0$, $\xi^{h-1} M \neq 0$.

Definition

Let $\mathcal{L}, \mathcal{L}'$ be equivalence classes of lattices in V. Then the distance $d(\mathcal{L}, \mathcal{L}') = ht(L/\xi^m L')$ where $L \in \mathcal{L}$, $L' \in \mathcal{L}'$, and m is as in §4.11. (This is well-defined by §4.12).

Note that if $r \in \mathbb{Z}$ with $\xi^r L' \subset L$ and if $L/\xi^r L' \sim (a_1, \ldots, a_n)$ with $a_1 \geq \ldots \geq a_n$ then $d(\mathcal{L}, \mathcal{L}') = a_1 - a_n$ (and $a_n = r-m$).

Lemma 4.13

Let $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ be equivalence classes of lattices in V.

(i) $d(\mathcal{L}, \mathcal{L}') = d(\mathcal{L}', \mathcal{L})$

(ii) $d(\mathcal{L}, \mathcal{L}') = 0$ iff $\mathcal{L} = \mathcal{L}'$.

(iii) $d(\mathcal{L}, \mathcal{L}') + d(\mathcal{L}', \mathcal{L}'') \geq d(\mathcal{L}, \mathcal{L}'')$

(iv) $d(\mathcal{L}, \mathcal{L}') = d \geq 2$ implies there exists $\mathcal{L}''$ such that $d(\mathcal{L}, \mathcal{L}'') = 1$, $d(\mathcal{L}'', \mathcal{L}') = d-1$.

Proof:

(i) Let $L \in \mathcal{L}$. By §4.11 there exists a unique $L' \in \mathcal{L}'$ such that $L' \subset L$ and $l(L/L') \leq n-1$; and then there exists a unique $m \in \mathbb{Z}$ such that $\xi^m L \subset L'$ and $l(L'/\xi^m L) \leq n-1$. Thus we have an exact sequence $0 \to L'/\xi^m L \to L/\xi^m L \to L/L' \to 0$.

Suppose $L'/\xi^m L \sim (a_1,\dots,a_{n-1})$ with $a_1 \geq \dots \geq a_{n-1}$. Then since $L/\xi^m L \sim (m,\dots,m)$ (length n if $m \neq 0$) we see $m \geq a_1$ and $L/L' \sim (m-a_1,\dots,m-a_{n-1},m)$; but $l(L/L') \leq n-1$ so $m-a_1 = 0$. So $d(\mathcal{L},\mathcal{L}') = ht(L/L') = m = a_1 = ht(L'/\xi^m L) = d(\mathcal{L}',\mathcal{L})$.

(ii) $d(\mathcal{L},\mathcal{L}') = 0$ iff $L/\xi^m L' = 0$, in the notation of §4.11, iff $L = \xi^m L'$ iff $\mathcal{L} = \mathcal{L}'$.

(iii) Given $L \in \mathcal{L}$ there exist unique $L' \in \mathcal{L}'$, $L'' \in \mathcal{L}''$ such that $L'' \subset L' \subset L$ and both L/L', L'/L'' have length $\leq n-1$. The exact sequence $0 \to L'/L'' \to L/L'' \to L/L' \to 0$ shows that $ht(L/L'') \leq ht(L'/L'') + ht(L/L') = d(\mathcal{L}',\mathcal{L}'') + d(\mathcal{L},\mathcal{L}')$.

But there exists a unique $m \in \mathbb{Z}$ such that $\xi^m L'' \subset L$ and $l(L/\xi^m L'') \leq n-1$, i.e. $l(\xi^{-m}L/L'') \leq n-1$. Now if $m > 0$ then $\xi^{-m}L/L'' \supsetneq L/L''$, so $l(L/L'') \leq n-1$ and the uniqueness of m is contradicted. So $m \leq 0$ and $L/L'' \supset \xi^{-m}L/L''$. Hence $d(\mathcal{L},\mathcal{L}'') = ht(\xi^{-m}L/L'') \leq ht(L/L'') \leq d(\mathcal{L}',\mathcal{L}'') + d(\mathcal{L},\mathcal{L}')$.

(iv) Given $L \in \mathcal{L}$ there exists a unique $L' \in \mathcal{L}'$ such that $L' \subset L$, and $l(L/L') \leq n-1$. Now there exists a bijection

$\{\text{lattices } L'' : L' \subset L'' \subset L\} \to \{\text{submodules of } L/L'\}$.

So choose L'' to correspond to $\xi(L/L')$.

Then $L/L'' \neq 0$ but $\xi(L/L'') = 0$, so $ht(L/L'') = 1$. Moreover $l(L/L'') \leq l(L/L') \leq n-1$, so if $\mathcal{L}''$ denotes the equivalence class of L'' we see $d(\mathcal{L},\mathcal{L}'') = 1$. But the exact sequence $0 \to L''/L' \to L/L' \to L/L'' \cong \frac{L/L'}{L''/L'} \to 0$ shows that $l(L''/L') \leq l(L/L') \leq n-1$, and $\xi^{d-1}(L''/L') = 0$. Finally, by part (iii) $ht(L''/L') \geq ht(L/L') - ht(L/L'') = d-1$, so $ht(L''/L') = d-1$, i.e. $d(\mathcal{L}'',\mathcal{L}') = d-1$. ■

We now define a simplicial complex $X(V)$ as follows. The vertices are equivalence classes of lattices in V, and for $0 \leq k \leq n-1$ the k+1 distinct vertices $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_k$ form a k-simplex iff $d(\mathcal{L}_i, \mathcal{L}_j) = 1$ $(0 \leq i < j \leq k)$.

Proposition 4.14

Let $0 \leq k \leq n-1$. Then the k+1 distinct vertices $\mathcal{L}_0, \ldots, \mathcal{L}_k$ form a k-simplex iff given $L_0 \in \mathcal{L}_0$ there exist unique $L_i \in \mathcal{L}_i$ $(1 \leq i \leq k)$ such that for some permutation σ of $\{1, 2, \ldots, k\}$ we have $\xi L_0 \subsetneq L_{\sigma(1)} \subsetneq L_{\sigma(2)} \subsetneq \cdots \subsetneq L_{\sigma(k)} \subsetneq L_0$.

Proof:

Suppose $\mathcal{L}_0, \ldots, \mathcal{L}_k$ do form a k-simplex; let $L_0 \in \mathcal{L}_0$. If $1 \leq i \leq k$ then there exists a unique $L_i \in \mathcal{L}_i$ such that $L_i \subset L_0$ and $l(L_0/L_i) \leq n-1$. But $d(\mathcal{L}_0, \mathcal{L}_i) = 1$ so $ht(L_0/L_i) = 1$; thus $\xi(L_0/L_i) = 0$, but $L_0/L_i \neq 0$ and so $\xi L_0 \subsetneq L_i \subsetneq L_0$.

Now suppose $1 \leq j \leq k$, $j \neq i$. Then there exists a unique $L_j \in \mathcal{L}_j$ such that $\xi L_0 \subsetneq L_j \subsetneq L_0$, and there exists a unique $m \in \mathbb{Z}$ such that $\xi^m L_j \subset L_i$ and $l(L_i/\xi^m L_j) \leq n-1$; also $ht(L_i/\xi^m L_j) = d(\mathcal{L}_i, \mathcal{L}_j) = 1$. Hence $\xi^m L_j \subsetneq L_i \subsetneq \xi^{m-1} L_j$.

So $\xi^{m+1} L_0 \subsetneq \xi^m L_j \subsetneq L_i \subsetneq \xi^{m-1} L_j \subsetneq \xi^{m-1} L_0$, and so $m=0$ or 1, i.e. either $L_j \subsetneq L_i \subsetneq \xi^{-1} L_j$ or $\xi L_j \subsetneq L_i \subsetneq L_j$.

This proves the implication $\Rightarrow$ by a suitable induction.

Conversely, given lattices $L_0, \ldots, L_k$ in V such that $\xi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k \subsetneq L_0$ then for $1 \leq i \leq k$ and $1 \leq i < j \leq k$ or $j=0$ L_j/L_i is a proper non-trivial subquotient of $L_0/\xi L_0 \sim (1^n)$; so $l(L_j/L_i) \leq n-1$ and $ht(L_j/L_i) = 1$. Hence $d(\mathcal{L}_j, \mathcal{L}_i) = 1$ where $\mathcal{L}_j$(resp. $\mathcal{L}_i$) is the equivalence class of L_j(resp. L_i).

This proves the implication $\Leftarrow$. ∎

Remark : since $l(L_0/\xi L_0) = n$ it only makes sense to discuss k-simplices for $0 \leq k \leq n-1$.

Proposition 4.15

For each vertex $\mathcal{L}$ of $X(V)$ $\operatorname{link}(\mathcal{L}) \cong T(\overline{V})$, the spherical Tits building associated to an $\mathcal{O}/\mathfrak{p}$ -vector space $\overline{V}$ of dimension n.

(Here $\operatorname{link}(\mathcal{L})$ is the subcomplex of $X(V)$ consisting of all simplexes whose vertices $\mathcal{L}'$ all satisfy $d(\mathcal{L},\mathcal{L}') = 1$.)

Thus the (k-1)-simplices of $\operatorname{link}(\mathcal{L})$ are precisely the (k-1)-faces opposite to $\mathcal{L}$ of the k-simplices of $X(V)$ which contain $\mathcal{L}$ $(1 \leq k \leq n-1)$.

Proof:

Fix $L \in \mathcal{L}$. Then $d(\mathcal{L},\mathcal{L}') = 1$ iff there exists a unique $L' \in \mathcal{L}'$ such that $\xi L \subsetneq L' \subsetneq L$, so the vertices of $\operatorname{link}(\mathcal{L})$ correspond bijectively to the proper non-zero subspaces $L'/\xi L$ of the $\mathcal{O}/\xi\mathcal{O}$-vector space $L/\xi L$ of dimension n. Also if $1 \leq k \leq n-1$:

the distinct vertices $\mathcal{L}_1,\ldots,\mathcal{L}_k$ of $\operatorname{link}(\mathcal{L})$ form a (k-1)-simplex

iff $\mathcal{L},\mathcal{L}_1,\ldots,\mathcal{L}_k$ form a k-simplex of $X(V)$

iff there exist unique $L_i \in \mathcal{L}_i$ $(1 \leq i \leq k)$ such that (after suitable re-ordering) $\xi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k \subsetneq L$

iff $0 \subsetneq L_1/\xi L \subsetneq L_2/\xi L \subsetneq \cdots \subsetneq L_k/\xi L \subsetneq L/\xi L$, which is a flag of distinct non-zero subspaces of $L/\xi L$

iff $L_1/\xi L, L_2/\xi L, \ldots, L_k/\xi L$ form a(k-1)-simplex of $T(L/\xi L)$. ∎

Remark

$X(V)$ is precisely the affine building associated by Bruhat and Tits ([3]) to the group $GL(V) \cong GL_n(K)$. In case $n=2$ $X(V)$ is an infinite tree in which each vertex is contained in precisely q+1 1-simplexes (cf. §4.15)

From now on fix a lattice L in V and an integer $h \geq 1$. Write $V = L/\xi^h L$ and (as in §1.0) $R = \mathcal{O}/\xi^h\mathcal{O}$. Let $\mathcal{L}$ denote the equivalence class of L, and denote by X(V) the subcomplex of X(V) spanned by those vertices $\mathcal{L}'$ such that $d(\mathcal{L}',\mathcal{L}) \leq h$.

Thus a vertex $\mathcal{L}'$ of X(V) lies in X(V) iff the unique $L' \in \mathcal{L}'$ such that $\xi^h L \subset L'$ and $l(L'/\xi^h L) \leq n-1$ also satisfies $ht(L'/\xi^h L) \leq h$, i.e. $\xi^h L \subset L' \subset L$. Now since the isomorphism type of $L'/\xi^h L$ is uniquely determined we see :

Corollary 4.16

The vertices of X(V) correspond bijectively to the submodules M of the free rank n R-module V, of length $\leq n-1$. ∎

In case h=1 X(V) is a spherical Tits building T(V) together with an extra vertex (corresponding to the zero submodule of V) which is joined to every other vertex (cf.§4.15). The simplices of T(V) correspond to flags of submodules of V.

4.2 Geometric description of St_G

Fix a flag $\mathfrak{F} = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V)$ of type (h^{n-1}). By §3.14, given $0 \leq i \leq m \leq h$ and $0 \leq r \leq n-2$ there exists a unique module $U \sim (m^r, i)$ such that $\pi^{h-m} V_r \subset U \subset \pi^{h-m} V_{r+1}$. So $\mathfrak{F}$ determines a subcomplex $C(\mathfrak{F})$ of X(V) which is spanned by the vertices U : there is precisely one for each isomorphism type of R-modules of length $\leq n-1$.

For each sequence $\mathfrak{k} = (k_1, \ldots, k_{n-1})$ of integers with $0 \leq k_r \leq h$ for each r, let $C(\mathfrak{F}, \mathfrak{k})$ denote the subcomplex of $C(\mathfrak{F})$ spanned by those vertices of $C(\mathfrak{F})$ at distance $\geq h-k_r$ from V_r (for each r). G acts transitively on $\mathcal{C} = \{C(\mathfrak{F}) : \mathfrak{F} \in \mathcal{F}(h^{n-1})\}$, by §1.11, and also on $\mathcal{C}(\mathfrak{k}) = \{C(\mathfrak{F}, \mathfrak{k}) : \mathfrak{F} \in \mathcal{F}(h^{n-1})\}$ since each $C \in \mathcal{C}$ contains a unique subcomplex $\pi_{\mathfrak{k}}(C) \in \mathcal{C}(\mathfrak{k})$.

Definition

We write $R_{\mathfrak{k}}$ for the permutation representation of G on $\mathcal{C}(\mathfrak{k})$.

Thus $R_{\mathbf{k}} = \{\text{functions } f:\mathcal{C}(\mathbf{k}) \to \mathbb{C}\}$

$$\cong \{f: \mathcal{C} \to \mathbb{C} \text{ such that } f(C) = f(C') \text{ whenever } \pi_{\mathbf{k}}(C) = \pi_{\mathbf{k}}(C')\},$$

identifying $R_{\mathbf{k}}$ as a subrepresentation of $R_{(h^{n-1})}$.

Proposition 4.21

$St_G \cong \sum_{\mathbf{k}}(-1)^{|\mathbf{k}|}R_{\mathbf{k}}$, the sum being taken over those $\mathbf{k}$ such that $((h-1)^{n-1}) \leq \mathbf{k} \leq (h^{n-1})$.

Proof:

If $((h-1)^{n-1}) \leq \mathbf{k} = (k_1,\ldots,k_{n-1}) \leq (h^{n-1})$ then $k_r = \begin{cases} h & r\in J \\ h-1 & r\notin J \end{cases}$ for some $J \subset S$. In view of §3.23 (and the fact that $|\mathbf{k}| = |J'|$) we must show that $R_{\mathbf{k}} \cong I(h^{n-1},J)$. But $I(\underline{a}) \cong 1_{B_{\underline{a}}}^G$ (by §3.17), and if $\underline{a} = (h^{n-1},J)$ then $B_{\underline{a}} = H_{\mathbf{k}}$. Thus we must show that $R_{\mathbf{k}} \cong 1_{H_{\mathbf{k}}}^G$. This is a special case of §4.31(i), and so we refer to the proof given there. ∎

Proposition 4.22

Write $R_G = \sum_{\mathbf{k}\leq(h^{n-1})}(-1)^{|\mathbf{k}|}R_{\mathbf{k}}$.

Then R_G is a subrepresentation of 1_B^G and contains St_G.

Proof:

If $\mathbf{k}, \boldsymbol{\ell}$ are sequences then $C(\mathcal{Y},\mathbf{k}) \cap C(\mathcal{Y},\boldsymbol{\ell})$ is the subcomplex of $C(\mathcal{Y})$ spanned by those vertices U of $C(\mathcal{Y})$ which satisfy both $d(U,V_r) \geq h-k_r$ and $d(U,V_r) \geq h-l_r$, i.e. $d(U,V_r) \geq h-\min(k_r,l_r)$, for each r, so its precisely $C(\mathcal{Y},\mathbf{k}\wedge\boldsymbol{\ell})$. Hence $\pi_{\mathbf{k}}(C) \cap \pi_{\boldsymbol{\ell}}(C) = \pi_{\mathbf{k}\wedge\boldsymbol{\ell}}(C)$ for each $C \in \mathcal{C}$.

Now if also $C' \in \mathcal{C}$ and $\pi_{\mathbf{k}}(C) = \pi_{\mathbf{k}}(C')$ or $\pi_{\boldsymbol{\ell}}(C) = \pi_{\boldsymbol{\ell}}(C')$ or both then $\pi_{\mathbf{k}\wedge\boldsymbol{\ell}}(C) = \pi_{\mathbf{k}\wedge\boldsymbol{\ell}}(C')$; and conversely given the latter equation then there exists $C'' \in \mathcal{C}$ with $\pi_{\mathbf{k}}(C) = C \cap C'' = \pi_{\mathbf{k}}(C'')$ and $\pi_{\boldsymbol{\ell}}(C'') = C'' \cap C' = \pi_{\boldsymbol{\ell}}(C')$, whence

$$\{f:\mathcal{C}\to \mathbb{C} : f(C) = f(C') \text{ whenever } \pi_{\mathbf{k}}(C) = \pi_{\mathbf{k}}(C') \text{ or } \pi_{\boldsymbol{\ell}}(C) = \pi_{\boldsymbol{\ell}}(C')\}$$

$= \{f : \mathcal{C} \to \mathbb{C} \ : \ f(c) = f(c') \text{ whenever } \pi_{k \wedge l}(c) = \pi_{k \wedge l}(c')\}$,

i.e. $R_k \cap R_l = R_{k \wedge l}$.

So applying §1.44(i) $R_G = CR_{(h^{n-1})}$ is a subrepresentation of $R_{(h^{n-1})} \cong 1_B^G$. —

Furthermore, using §4.21, $R_G = St_G + \sum_{m=1}^{n-1} S_m$ ----(*),

where $S_m = \sum_{k \in L_m} (-1)^{|k|} R_k$; here L_m is the set

$\{k = (k_1,\ldots,k_{n-1}) \leq (h^{n-1-m}, h-2, h^{m-1})$ such that $k_r \geq h-1$ for $n-m+1 \leq r \leq n-1\}$,

and satisfies axioms (1) and (2) for §1.44, so S_m is a subrepresentation of $R_{(h^{n-1-m}, h-2, h^{m-1})}$ and in particular S_m is 'positive' (lies in M^+ in the notation of §1.44).

So by (*) St_G is a subrepresentation of R_G. ∎

4.3 Geometric description of S_G

For $0 \leq m \leq h$ write $K_m = \{k = (k_1,\ldots,k_{n-1}) : m = \max\{k_r : 1 \leq r \leq n-1\}\}$. If $k \in K_m$ then let $D(\mathcal{F},k)$ be the subcomplex of $C(\mathcal{F},k)$ which is spanned by those vertices at distance $\leq m$ from 0, and let Q_k be the permutation representation of G on $\{D(\mathcal{F},k) : \mathcal{F} \in \mathfrak{F}(h^{n-1})\}$. Then we identify Q_k as a subrepresentation of R_k via $Q_k \cong \{f : \mathcal{C}(k) \to \mathbb{C}$ such that $f(C(\mathcal{F},k)) = f(C(\mathcal{F}',k))$ whenever $D(\mathcal{F},k) = D(\mathcal{F}',k)\}$.

For the following result assume $\mathcal{F} = \mathcal{F}_S$ is the standard flag, so $V_i = \langle e_1,\ldots,e_i \rangle$ for each i. With respect to the basis $\underset{\sim}{e}$ we have defined the matrix group H_k (§2.1).

Proposition 4.31

(i) $\mathrm{Stab}_G(D(\mathcal{F},k)) = H_k$

(ii) $S_G = \sum_{k \leq (h^{n-1})} (-1)^{|k|} Q_k$.

Proof:

(i) For convenience we suppress reference to $\mathcal{F}$ in this proof. First suppose $m = h$; so $D(k) = C(k)$. Let $0 < l < h$ and consider the vertices of $C(\mathfrak{J})$ at distance l from V_r : they are determined by the $(n-1)$ vertices

$$\begin{cases} < e_1,\ldots,e_{s-1},\pi^l e_s,\ldots,\pi^l e_r > , & 1\leq s\leq r \\ < e_1,\ldots,e_r,\pi^{h-l} e_{r+1},\ldots,\pi^{h-l} e_s > , & r+1\leq s\leq n-1. \end{cases}$$

This follows in the same way that $\mathfrak{J}$ determines $C(\mathfrak{J})$ (and in particular the vertices of $C(\mathfrak{J})$ at distance h from 0), because of the homogeneous nature of the construction of $X(\mathcal{V})$.

Thus $g \in G$ fixes $C(\mathfrak{k})$ provided it fixes these n-1 vertices, with $l=h-k_r$, for each r. Let $g = (m_{ij})$; then g fixes these vertices iff $\pi^{h-l}|m_{ij}$ for $\alpha_{ij} \in \Phi^- - \Phi^-_{\{r\}'}$, i.e. g has the form $\begin{pmatrix} * & * \\ A & * \end{pmatrix}$ where $A = (a_{ij})$ is an $(n-r)\times r$ matrix such that $\pi^{h-l}|a_{ij}$.

Write $X_l = \{r\in S : l = h-k_r\}$ and $S_{h-l} = X_0 \cup X_1 \cup .. \cup X_l$.
Then g fixes $C(\mathfrak{k})$ iff $\pi^{h-l}|m_{ij}$ for $\alpha_{ij}\in \Phi^- - \Phi^-_{X_l'}$ for each l

iff $\pi^{h-l}|m_{ij}$ for $\alpha_{ij}\in \Phi^- - \Phi^-_{S'_{h-l}}$ for each l

iff $\pi^{k_{ij}}|m_{ij}$ for $1\leq j<i\leq n$,

where $k_{i+1,i} = k_i$ (since $i \in X_{h-k_i}$) and $k_{ij} = \max\{k_r : j\leq r<i\}$.
This is because $\alpha_{ij}\in \Phi^-_Y$ iff $Y \supset \{j,j+1,\ldots,i-1\}$, and so α_{ij} lies outside $\Phi^-_{S'_{h-l}}$ iff there exists r with $j\leq r<i$ and $r \notin S'_{h-l}$, i.e. $l \geq h-k_r$, i.e. $k_r \geq h-l$.

This proves that $H_{\mathfrak{k}} = \mathrm{Stab}_G D(\mathfrak{k})$ in case m=h.

In case $m < h$ we have $\mathrm{Stab}_G D(\mathfrak{k}) = \pi_m^{-1}(\mathrm{Stab}_{G(m)} D(\mathfrak{k})) = \pi_m^{-1}\{(m_{ij}) \in G(m) : \pi^{l_{ij}}|m_{ij}$ for $1\leq j<i\leq n\}$, where $l_{i+1,i} = m-k_i$, $l_{ij} = \max\{l_{r+1,r} : j\leq r<i\}$, by the result already proved (with m instead of h). So $\mathrm{Stab}_G D(\mathfrak{k}) = H_{\mathfrak{k}}$, in any case; thus (i) is proved.

(ii) follows directly from (i) and §2.14. ∎

Theorem 4.32

(i) S_G is a subrepresentation of 1_B^G.

(ii) St_G is contained in S_G provided that at least one of the following hypotheses holds : (a) $q \geq 3$; (b) $h=2$; (c) $n=2$.

Proof:

The method of proof is to relate S_G to R_G and then use Proposition 4.22. Note that if $n=2$ then $S_G = R_G$ and the result is immediate. Part (ii) is probably true quite generally : the limitation of our proof lies in the dimension estimate of part (iii) of the lemma below.

For $1 \leq m \leq h-1$ and $k \in K_m$ define $P_k = R_k - Q_k$ (this is a subrepresentation of R_k), and $R_m = \varepsilon^{h-m} \sum_{k \in K_m} (-1)^{|k|} P_k$, where $\varepsilon = (-1)^{|(h-1)^{n-1}|} = (-1)^{n-1}$.

Lemma 4.33

(i) $R_G = S_G + \sum_{m=1}^{h-1} \varepsilon^{h-m} R_m$

(ii) R_m is a subrepresentation of $P_{(m^{n-1})}$ (hence of $R_{(m^{n-1})}$)

(iii) $\sum_{m=1}^{h-1} \dim(R_m) < \dim(St_G)$, provided $q \geq 3$ or $h=2$ (or both).

Proof:

(i) Note that if $m = 0$ or h then $R_k = Q_k$. The result is now immediate from §4.31(ii) and the definitions of R_G and R_m.

(ii) Let $k, l \in K_m$. Then $C(\mathfrak{z},k) \cap D(\mathfrak{z},l) = D(\mathfrak{z},k) \cap D(\mathfrak{z},l) = D(\mathfrak{z},k) \cap C(\mathfrak{z},l)$, so $R_k \cap Q_l = Q_k \cap R_l = Q_k \cap Q_l = Q_{k \cap l}$, and so

$$\begin{aligned} P_k \cap P_l &= (R_k - Q_k) \cap (R_l - Q_l) \\ &= R_k \cap R_l - Q_k \cap R_l - R_k \cap Q_l + Q_k \cap Q_l \\ &= R_{k \cap l} - Q_{k \cap l}. \end{aligned}$$

Assuming m fixed, extend the definition of P_k to all $k \leq (m^{n-1})$ by requiring $P_{k_1} \cap P_{k_2} = P_{k_1 \cap k_2}$ (note that any such k has the form $k_1 \cap k_2$ for some $k_1, k_2 \in K_m$). Now $P_{(m^{n-1})} = \sum_{k \leq (m^{n-1})} CP_k$ by §1.44(ii) so if $\varepsilon = 1$ then $R_m + CP_{((m-1)^{n-1})} = CP_{(m^{n-1})}$ and if $\varepsilon = -1$ then

$R_m = CP_{((m-1)^{n-1})} + CP_{(m^{n-1})}$, so in either case R_m is a subrepresentation of $P_{(m^{n-1})}$.

(iii) $\mathrm{Stab}_G C(m^{n-1}) = \langle B, U_{i+1,i}(\pi^m R) : 1 \leq i \leq n-1 \rangle = B(m)$, say, and $|B(m) : B| = q^{r_m}$ with $r_m \geq m(n-1)$.

Hence
$$\begin{aligned}\sum_{m=1}^{h-1} \dim(R_m) &\leq \sum_{m=1}^{h-1} \dim(R_{(m^{n-1})}) \quad \text{by part (ii)}\\ &= \sum_{m=1}^{h-1} |G:B|.q^{-r_m}\\ &\leq |G:B| \sum_{m=1}^{h-1} q^{-m(n-1)}\\ &< |G:B|.q^{-(n-1)}.\sum_{i=0}^{\infty} q^{-i}\\ &= |G:B|.q^{-(n-1)}.q/(q-1).\end{aligned}$$

But if $q \geq 3$ then $(q-1)^2-q = (q-3/2)^2-5/4 > 0$. So we have $q/(q-1) < q-1 \leq (q-1)^{n-1}$ $(n \geq 2)$, and hence
$$\sum_{m=1}^{h-1} \dim(R_m) < |G:B|.q^{-(n-1)}.(q-1)^{n-1} = \dim(St_G) \quad (\text{by } \S 3.24).$$

If h=2 then
$$\begin{aligned}\sum_{m=1}^{h-1} \dim(R_m) &\leq \dim(P_{(1^{n-1})}) \quad \text{by part (ii)}\\ &< \dim(R_{(1^{n-1})})\\ &= |G:B|.q^{-(n-1)}\\ &\leq |G:B|.q^{-(n-1)}(q-1)^{n-1} = \dim(St_G).\end{aligned}$$
∎

Remark: by using the full strength of part (ii) one can prove part (iii) in case q=2 for 'small enough h', but a more accurate estimate of $\dim(R_m)$ would be required to prove it in general.

To deduce the theorem we consider two cases :

$\varepsilon = 1$ (n odd): $R_G = S_G + \sum_{m=1}^{h-1} R_m$, which is a sum of 'positive' representations, so by §4.22 S_G is a subrepresentation of 1_B^G; and furthermore, R_G contains St_G. But St_G is irreducible of dimension $> \dim(\sum_{m=1}^{h-1} R_m)$ so must be contained in S_G.

$\underline{\varepsilon = -1}$ (n even) : $\quad S_G \subset S_G + \sum\limits_{\substack{m=1 \\ (h-m)\text{even}}}^{h-1} R_m = R_G + \sum\limits_{\substack{m=1 \\ (h-m)\text{odd}}}^{h-1} R_m$,

and this equals $R_G + \sum\limits_{\substack{m=1 \\ (h-m)\text{odd}}}^{h-1} (CP_{(m^{n-1})} + CP_{((m-1)^{n-1})})$, by the

proof of §4.33(ii), i.e. $R_G + \sum\limits_{m=j}^{h-1} CP_{(m^{n-1})}$ where $j = \begin{cases} 1 & h \text{ odd} \\ 0 & h \text{ even} \end{cases}$.

But $R_G = CR_{(h^{n-1})}$ by definition; and also $Q_k = R_k - P_k$, so $CR_k - CP_k = CQ_k$ which is 'positive' by §1.44(i), and so $CP_k \subset CR_k$, for any k . Hence $S_G \subset \sum\limits_{m=j}^{h} CR_{(m^{n-1})}$ and this is contained in $\sum\limits_{k \leq (h^{n-1})} CR_k$, which equals $R_{(h^{n-1})} \cong 1_B^G$, by §1.44(ii). Thus $S_G \subset 1_B^G$.

Finally, St_G is contained in R_G and hence in $S_G + \sum\limits_{\substack{m=1 \\ (h-m)\text{even}}}^{h-1} R_m$,

but is also irreducible of dimension greater than

$\sum\limits_{m=1}^{h-1} \dim(R_m)$, which is greater than $\dim(\sum\limits_{\substack{m=1 \\ (h-m)\text{even}}}^{h-1} R_m)$.

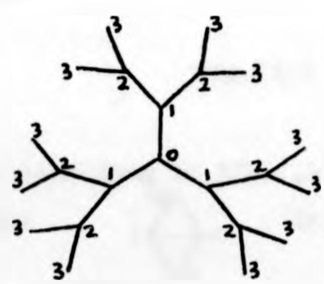
So St_G must be contained in S_G. ∎

4.4 Examples

Case n=2

X(V) is a finite tree : we show the case when q=2,h=3. The type of the module corresponding to each vertex is indicated.

fig. (i)



Each vertex is contained in q+1 1-simplexes, except those of type (h), which are contained in a unique 1-simplex.

I(i) is isomorphic to the permutation representation of G on the set of vertices of X(V) of type (i).

From §4.2/3 we have :

$$S_G \cong \sum_{i=0}^{h} (-1)^{h-i} I(i)$$

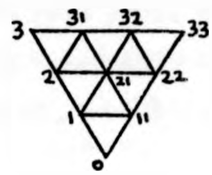
$$St_G \cong I(h) - I(h-1) ,$$

and this latter formula is equally valid for h=1 and for h≥2.

<u>Case n=3</u>

We show $C(\mathcal{Y})$ in case h=3, again labelling the vertices with the type of the corresponding module.

fig.(ii)



I(h,j) is isomorphic to the permutation representation of G on the set of subcomplexes of X(V) of type shown in figure (iii), which is just  in case h=1 and j=0. (this follows from §4.31). The complex corresponding to I(j,h) is the one laterally symmetric to this.

fig.(iii)

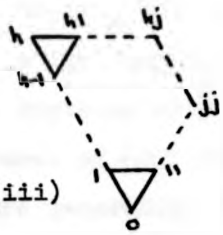
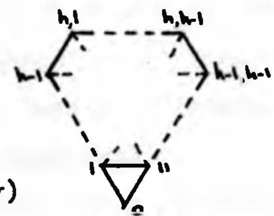


fig.(iv)



The complex corresponding to I(h-1,h-1) is shown in figure (iv) and is just  in case h=2, and is just  in case h=1.

We have $S_G \cong I(h,h) + \sum_{i=0}^{h-1} (-1)^{h-i}(I(h,i)+I(i,h)) + S_{G(h-1)}$,

$$St_G \cong I(h,h) - I(h,h-1) - I(h-1,h) + I(h-1,h-1),$$

and this latter formula is equally valid for h=1 and for h≥2.

4.5 Constructions for more general groups

Let $\underline{\underline{G}}$ be a connected semisimple linear algebraic group defined over K. Then $\underline{\underline{G}}(K)$ has the structure of Tits system with (affine) Weyl group $\hat{W}$, say ([12]) and associated to this system there is a building $\mathfrak{I}$ on which G(K) acts ([3]).

Every chamber of $\mathfrak{I}$ contains at least one special point p; in case $\underline{\underline{G}}$ is simply connected the maximal parahoric subgroups of $\underline{\underline{G}}(K)$ are precisely the maximal compact subgroups so the stabiliser of p is a special maximal compact subgroup P of $\underline{\underline{G}}(K)$, i.e. one of the form $\underline{\underline{G}}(\mathcal{O})$, which may be defined as a Chevalley group using a suitable maximal split torus $\underline{\underline{A}}$ of $\underline{\underline{G}}$ and Chevalley system $(x_\alpha : \alpha \in \Phi)$ of $(\underline{\underline{G}},\underline{\underline{A}})$ (see [9],§4.1).

$\underline{\underline{G}}(\mathcal{O})$ has subgroups $\underline{\underline{G}}(\mathfrak{p}^h)$ $(h \geq 1)$ ([9],§2.2.7) and $\underline{\underline{G}}(\mathcal{O})$ is the projective limit of the finite groups $G_h = \underline{\underline{G}}(\mathcal{O})/\underline{\underline{G}}(\mathfrak{p}^h)$. G_h acts on the subcomplex X_h of $\mathfrak{I}$ stabilised by $\underline{\underline{G}}(\mathfrak{p}^h)$.

Now each apartment A of $\mathfrak{I}$ is (isomorphic to) the Coxeter complex of $\hat{W}$ and may be constructed as a space of valuations of $(x_\alpha : \alpha \in \Phi)$ ([3],§6.2.6,§6.5,§7.4); this description shows that $X_h \cap A$ is bounded by the walls of the affine roots $a_{\alpha,h}$ $(\alpha \in \Phi)$ and X_h is just the subcomplex of $\mathfrak{I}$ spanned by those vertices of $\mathfrak{I}$ at 'distance' $\leq$ h from p ('distance' in the sense of §4.1 along a path of 1-simplices).

More generally, the same results hold for $\underline{\underline{G}}$ reductive with simply connected derived group ([9],§4.1).

So in case $G_h = GL(V)$ we have $X_h \approx X(V)$ and the subcomplex C of $X_h \cap A$ bounded by the walls $\partial a_{\alpha,0}$ of the affine roots $a_{\alpha,0}$ $(\alpha \in \Sigma = \{\text{simple roots of } \Phi\})$ plays the role of the complex $C(\mathfrak{I})$ of §4.2. For each $\beta \in \Sigma$, $\bigcap_{\beta \neq \alpha \in \Sigma} \partial a_{\alpha,0}$ contains a unique vertex at distance h from p and these l vertices play the role of the vertices $V_1,\ldots,V_{n-1}$ of $C(\mathfrak{Y})$ $(l = |\Sigma|)$.

We can now proceed exactly as in §4.2/3 to construct subcomplexes $C(\mathbf{k})$, $D(\mathbf{k})$ of C for each sequence $\mathbf{k} = (k_1,\ldots,k_l)$ of integers with $0 \leq k_r \leq h$ $(1 \leq r \leq l)$; and then we can make the definitions of St_{G_h} and S_{G_h} corresponding to §4.21 and §4.31.

Alternatively we may use the description of G_h as a Chevalley group to construct parabolic and Levi subgroups of G_h, and then make a definition of S_{G_h} analogous to §2.12; or we may define a subgroup $H_{\mathcal{S}}$ of G_h for each filtration $\mathcal{S} = (\emptyset = S_{h+1} \subset \ldots \subset S_0 = \Sigma)$ of Σ (cf. §2.13/4) and define S_{G_h} as an alternating sum of representations of the form $1_{H_{\mathcal{S}}}^{G_h}$ as in §2.14, and $St_G = \sum_{X \subset \Sigma} (-1)^{|X|} 1_{H_{\mathcal{S}_X}}^{G_h}$ where $\mathcal{S}_X = (\emptyset \subset X' \subset \Sigma \subset \ldots \subset \Sigma)$.
If $h \geq 2$ then $H_{\mathcal{S}_X} = \langle \underline{A}(R), x_\alpha(R) \ (\alpha \in \Phi^+), x_{-\alpha}(\pi^{h-1}R) \ (\alpha \in X) \rangle$, and writing $B = H_{\mathcal{S}_\emptyset}$ we have $|H_{\mathcal{S}_X} : B| = q^{|X|}$ (by §1.34 and §1.56). In case $G_h = GL_n(R)$ we have $H_{\mathcal{S}_X} = \{(a_{ij}) \in GL_n(R) : a_{ij} = 0$ if $i > j$ and $-\alpha_{ij} \notin X;\ \pi^{h-1} | a_{ij}$ if $-\alpha_{ij} \in X\}$. (cf. §2.13).
Now if $\underline{G}$ is semisimple then $\underline{A}(R) \cong R^* \times \ldots \times R^*$ (l copies) and so $|G_h : H_{\mathcal{S}_X}| = q^{|X'| - lh} . |G_h| / |U| . (q-1)^l$, where $U = \langle x_\alpha(R) : \alpha \in \Phi^+ \rangle$. But by [18], §9 and [9], §2.2.5/7, $|G_h| = |G_1| . |\underline{G}(\mathfrak{p}) : \underline{G}(\mathfrak{p}^h)| = q^N \prod_{i=1}^{l} (q^{d_i} - 1) . q^{(l+2N)(h-1)}$ where d_i are the degrees of the Weyl group of Φ and $N = |\Phi^+|$. So for $h \geq 2$ $\dim(St_{G_h}) = |G_h| / |U| q^{lh} = q^{Nh - l - N} \prod_{i=1}^{l} (q^{d_i} - 1)$; note that $\sum_{i=1}^{l} d_i = l + N$ so the leading term of the polynomial $\dim(St_{G_h})$ is $q^{Nh} = \dim(S_{G_h})$.

With appropriate modification of notation the proof of §3.25 goes through, showing that St_{G_h} is irreducible. It is notable that in case h=1 G_1 does not appear to have any irreducible representations of dimension $|G_1|/q^k$ (k an integer) in contrast to the existence of St_{G_h} of dimension $|G_h|/q^{(N+1)h}$ when $h \geq 2$.

Chapter 5

Complements on GL_2 and GL_3

This final chapter consists of examples and counterexamples. We begin by considering two aspects of the structure of G which are significantly more complicated in general than in case R is a field (h=1). We show that if n=3 and $h \geq 3$ then both the number of orbits of B on $\mathcal{F}_S$ (or equivalently the double cosets of B in G) and the number of unipotent conjugacy classes of G are not bounded independently of q; moreover, there is no simple geometric characterisation of the B-orbits on $\mathcal{F}_S$. The cases when n=2 and when n=3,h=2 are misleadingly simple.

Next we decompose the representations 1_B^G and S_G in the cases n=2 and n=3,h=2; the number of distinct irreducible components of 1_B^G equals the number of unipotent conjugacy classes of G in these cases (as in case h=1).

In §5.4 we give an example, in case n=3,h=3, of an element of G at which the character of S_G is neither 0 nor $\pm$ a power of q. Finally, in §5.5 we compute the character of St_G at all split semisimple elements of G in the cases n=2,3 leading to a conjecture for the cases $n \geq 4$.

5.1 B-B double cosets

Let $\rho_S = (0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V) \in \mathcal{F}_S$ again denote the standard flag of type S in V.

Lemma 5.11

A necessary condition for $\sigma = (0 \subset U_1 \subset \ldots \subset U_{n-1} \subset V)$ and $\sigma' = (0 \subset U_1' \subset \ldots \subset U_{n-1}' \subset V) \in \mathcal{F}_S$ to lie in the same B-orbit of $\mathcal{F}_S$ is $U_i \cap V_j \cong U_i' \cap V_j \quad (1 \leq i, j \leq n-1)$.

Proof:

If there exists $b \in B$ such that $b.\sigma = \sigma'$ then $b.U_i = U_i'$ $(1 \leq i \leq n-1)$ so for $1 \leq i, j \leq n-1$ we have $U_i' \cap V_j = bU_i \cap V_j = b(U_i \cap V_j)$, since B stabilises ρ_S, and this is isomorphic to $U_i \cap V_j$. ∎

Now in case h=1 the condition $U_i \cap V_j \cong U_i' \cap V_j$ $(1 \leq i, j \leq n-1)$ is also sufficient to ensure that σ, σ' lie in the same orbit of B (§1.47). This condition is sufficient also in cases $n=2, h \geq 2$, as follows from the fact that

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \quad : 1 \leq i \leq h \right\}$$

forms a complete set of representatives for distinct B-B cosets in G ([16],p60) ; the possible isomorphism types for $U_1 \cap V_1$ are (i) for $0 \leq i \leq h$.

Proposition 5.12

Let $n=3, h \geq 3$. Let $r, r' \in R^*$ such that $r-r', r-1$ and $r'-1$ all lie outside πR. Let $g = \begin{pmatrix} 1 & 0 & 0 \\ \pi & r & 0 \\ \pi^2 & \pi & 1 \end{pmatrix}$, $g' = \begin{pmatrix} 1 & 0 & 0 \\ \pi & r' & 0 \\ \pi^2 & \pi & 1 \end{pmatrix}$, and let $g.\rho_S = (0 \subset U_1 \subset U_2 \subset V)$, $g'.\rho_S = (0 \subset U_1' \subset U_2' \subset V)$.

Then (i) $U_i \cap V_j \cong U_i' \cap V_j$ $(1 \leq i, j \leq n-1)$

(ii) $g.\rho_S$, $g'.\rho_S$ do not lie in the same B-orbit

(iii) the number of B-orbits on $\mathcal{F}_S$ is not bounded independently of q.

Proof:

(i) $U_1 = \langle g.e_1 \rangle = \langle e_1 + \pi e_2 + \pi^2 e_3 \rangle = \langle g'.e_1 \rangle = U_1'$, so $U_1 \cap V_j = U_1' \cap V_j$ $(j=1,2)$.

$U_2 = \langle e_1 + \pi e_2 + \pi^2 e_3, re_2 + \pi e_3 \rangle$, $U_2' = \langle e_1 + \pi e_2 + \pi^2 e_3, r'e_2 + \pi e_3 \rangle$, so $U_2 \cap V_1 = \langle \pi^{h-1} e_1 \rangle = U_2' \cap V_1$ (since $\pi(r-1), \pi(r'-1) \notin \pi^2 R$).

Finally, $U_2 \cap V_2 = \langle e_1 + \pi(1-r)e_2, \pi^{h-1} re_2 \rangle$
$\cong \langle e_1 + \pi(1-r')e_2, \pi^{h-1} r'e_2 \rangle = U_2' \cap V_2$.

(ii) If g, g' were in the same B-B coset in G we would have an equation $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \pi & r & 0 \\ \pi^2 & \pi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \pi & r' & 0 \\ \pi^2 & \pi & 1 \end{pmatrix}\begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}$

so in particular $\begin{cases} f\pi^2 = \pi^2 a' \ , \ dr + e\pi = \pi b' + r'd' \ , \\ \pi f = \pi^2 b' + \pi d' \ , \ d\pi + e\pi^2 = a'\pi . \end{cases}$

Hence $r'd'\pi^{h-1} = dr\pi^{h-1} = ra'\pi^{h-1} = rf\pi^{h-1}$, since $h-1 \geq 2$, and this equals $rd'\pi^{h-1}$. So $(r-r')\pi^{h-1} = 0$, since $d' \in R^*$, which contradicts the assumption $r-r' \notin \pi R$.

(iii) is a consequence of (ii). ■

5.2 Unipotent conjugacy classes

If $n=2$ then G has $h+1$ unipotent conjugacy classes, with representatives $\begin{pmatrix} 1 & \pi^i \\ 0 & 1 \end{pmatrix}$ $(0 \leq i \leq h)$ for example ([16],p.101). If $n=3$ the situation is more complicated; there are two features of particular interest.

Proposition 5.21

(i) $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$ are conjugate if and only if either $a \| b$ or one (or both) of a, b is zero.

(ii) If $h \geq 3$ then the $(q-1)$ elements $\begin{pmatrix} 1 & \pi & \tilde{r} \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix}$ $(r \in \bar{R}^*)$ lie in distinct conjugacy classes.

Proof:

First note $\begin{pmatrix} k^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} 1 & k\pi^r & f\pi^s \\ 0 & 1 & m\pi^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \pi^r & e\pi^s \\ 0 & 1 & \pi^t \\ 0 & 0 & 1 \end{pmatrix}$

so assume that $a = \pi^r$, $b = \pi^t$. We shall show in fact that $\begin{pmatrix} 1 & \pi^r & 0 \\ 0 & 1 & \pi^t \\ 0 & 0 & 1 \end{pmatrix}$ $(0 \leq r, t \leq h)$ lie in distinct conjugacy classes and that $\begin{pmatrix} 1 & \pi^r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(0 \leq r \leq h)$ give $h+1$ further classes.

So let $0 \leq r,t,l,m \leq h$. Then the equation

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & j & k \end{pmatrix}\begin{pmatrix} 1 & \pi^r & 0 \\ 0 & 1 & \pi^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pi^l & 0 \\ 0 & 1 & \pi^m \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & j & k \end{pmatrix} \quad \text{implies}$$

$$\begin{cases} 0 = \pi^t j = \pi^r g = \pi^m g = \pi^l d \\ \pi^r a = \pi^l e \ , \ \pi^m k = \pi^t e \\ \pi^r d = \pi^m j \ , \ \pi^t b = \pi^l f \end{cases} \quad . \text{ Write } \gamma = \begin{pmatrix} a & b & c \\ d & e & f \\ g & j & k \end{pmatrix} .$$

Case 1 : $l < h$, $m < h$. Then $\pi | g,d$. If $t=h$ then $\pi | k,f$ so $\pi | \det(\gamma)$, a contradiction. So $t < h$. Hence $\pi | j$, and so $\pi \nmid a,e,k$ since $\pi \nmid \det(\gamma)$. Hence $l=r$ and $m=t$.

Case 2 : $h=l > m$. Then $\pi | g$. If both $r < h$ and $t < h$ then $\pi | g,j,b,a$, whence $\pi | \det(\gamma)$, a contradiction. If $r=h$ then $\pi | j$ so since $\pi | g$ and $\pi \nmid \det(\gamma)$ we have $\pi \nmid k$. Hence $t \leq m < h$, and so $\pi | b$. But also $\pi | j$ and $\pi \nmid \det(\gamma)$ so $\pi \nmid e$. Thus $\pi \nmid e,k$ and so $m=t$. If $t=h$ instead then a similar argument shows that $m=r$.

Case 3 : $h=l=m$. Then $\begin{pmatrix} 1 & \pi^l & 0 \\ 0 & 1 & \pi^m \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is conjugate only to itself; so $r=t=h$.

So in any case we have either $l=r, m=t$ or $l=h=t, m=r$ or $m=h=r, l=t$. To complete the proof note that for any $x \in R$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} .$$

(ii) The equation $\begin{pmatrix} a & b & c \\ d & e & f \\ g & j & k \end{pmatrix}\begin{pmatrix} 1 & \pi & \tilde{r}' \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pi & \tilde{r} \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & j & k \end{pmatrix}$

implies $\begin{cases} d\pi = j\pi \ , \ g\pi = 0 \ , \ d\pi + \tilde{r}g = 0 = j\pi + \tilde{r}'g \ , \\ k\pi = e\pi + \tilde{r}'d \ , \ a\pi = e\pi + \tilde{r}j \ , \\ \tilde{r}'a + b\pi = \tilde{r}k + f\pi \ . \end{cases}$ Suppose $r \neq r'$.

Thus $(\tilde{r}-\tilde{r}')g = 0$; so since $\pi \nmid (\tilde{r}-\tilde{r}')$ we see $g = 0$. Also, $\pi | d,j$ so since $\pi \nmid \det(\gamma)$ we have $\pi \nmid a,e,k$. But $\cancel{k\pi}^{h-1} = e\pi^{h-1} = a\pi^{h-1}$, and $\tilde{r}'a\pi^{h-1} = \tilde{r}k\pi^{h-1}$. So $(\tilde{r}-\tilde{r}')\pi^{h-1} = 0$, contradiction. ■

In case n=3,h=2 the number of unipotent conjugacy classes of G is independent of q.

<u>Proposition 5.22</u>

The following 7 elements provide a complete set of representatives for distinct conjugacy classes of unipotent elements of G in the case n=3,h=2.

$$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&\pi\\0&0&1\end{pmatrix},\begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}1&\pi&0\\0&1&\pi\\0&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&\pi&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

<u>Proof</u>:

By the proof of §5.21(i) it is sufficient to consider elements of the form $\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}$ ($\lambda\in R^*$, $0\leq r,s,t\leq h$).

If $r\leq s$ then $\begin{pmatrix}1&0&0\\0&1&\lambda\pi^{s-r}\\0&0&1\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-\lambda\pi^{s-r}\\0&0&1\end{pmatrix}=\begin{pmatrix}1&\pi^r&0\\0&1&\pi^t\\0&0&1\end{pmatrix}$.

If $t\leq s$ then $\begin{pmatrix}1&-\lambda\pi^{s-t}&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}\begin{pmatrix}1&\lambda\pi^{s-t}&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&\pi^r&0\\0&1&\pi^t\\0&0&1\end{pmatrix}$.

If $r+t\geq h+s$ then $\pi^{r+t-s}=0$ and so if also $s<r$ and $s<t$

$$\begin{pmatrix}1&0&0\\0&\pi^{r-s}&\lambda\\-\lambda^{-1}\pi^{t-s}&1&0\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}=\begin{pmatrix}1&\pi^s&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\pi^{r-s}&\lambda\\-\lambda^{-1}\pi^{t-s}&1&0\end{pmatrix}.$$

This covers all cases if h=2 since if $s<r$ and $s<t$ then $r\geq 1$ and $t-s\geq 1$ so $r+t-s\geq 2=h$. The result now follows from (the proof of) §5.21(i). ■

<u>5.3 Decomposition of 1_B^G and S_G</u>

<u>Proposition 5.31</u>

Assume that n=2. Then $1_B^G\cong\bigoplus_{i=0}^{h}St_{G(i)}$ (writing $St_{G(0)}=1$),

and $S_G\cong St_G\oplus St_{G(h-2)}\oplus\cdots\oplus\begin{cases}St_{G(2)}\oplus St_{G(0)} & \text{(h even)}\\ St_{G(3)}\oplus St_{G(1)} & \text{(h odd)}.\end{cases}$

Also $1_B^G\cong S_G\oplus S_{G(h-1)}$.

In case $n=3, h=2$ the number of unipotent conjugacy classes of G is independent of q.

<u>Proposition 5.22</u>

The following 7 elements provide a complete set of representatives for distinct conjugacy classes of unipotent elements of G in the case $n=3, h=2$.

$$\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&\pi\\0&0&1\end{pmatrix}, \begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&\pi&0\\0&1&\pi\\0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&\pi&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

<u>Proof</u>:

By the proof of §5.21(i) it is sufficient to consider elements of the form $\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}$ $(\lambda \in R^*,\ 0 \le r,s,t \le h)$.

If $r \le s$ then $\begin{pmatrix}1&0&0\\0&1&\lambda\pi^{s-r}\\0&0&1\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-\lambda\pi^{s-r}\\0&0&1\end{pmatrix} = \begin{pmatrix}1&\pi^r&0\\0&1&\pi^t\\0&0&1\end{pmatrix}$.

If $t \le s$ then $\begin{pmatrix}1&-\lambda\pi^{s-t}&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix}\begin{pmatrix}1&\lambda\pi^{s-t}&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&\pi^r&0\\0&1&\pi^t\\0&0&1\end{pmatrix}$.

If $r+t \geq h+s$ then $\pi^{r+t-s} = 0$ and so if also $s < r$ and $s < t$

$$\begin{pmatrix}1&0&0\\0&\pi^{r-s}&\lambda\\-\lambda^{-1}\pi^{t-s}&1&0\end{pmatrix}\begin{pmatrix}1&\pi^r&\lambda\pi^s\\0&1&\pi^t\\0&0&1\end{pmatrix} = \begin{pmatrix}1&\pi^s&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\pi^{r-s}&\lambda\\-\lambda^{-1}\pi^{t-s}&1&0\end{pmatrix}.$$

This covers all cases if $h=2$ since if $s < r$ and $s < t$ then $r \geq 1$ and $t-s \geq 1$ so $r+t-s \geq 2 = h$. The result now follows from (the proof of) §5.21(i). ∎

<u>5.3 Decomposition of 1_B^G and S_G</u>

<u>Proposition 5.31</u>

Assume that $n=2$. Then $1_B^G \cong \bigoplus_{i=0}^{h} St_{G(i)}$ (writing $St_{G(0)} = 1$), and $S_G \cong St_G \oplus St_{G(h-2)} \oplus \dots \oplus \begin{cases} St_{G(2)} \oplus St_{G(0)} & (h \text{ even}) \\ St_{G(3)} \oplus St_{G(1)} & (h \text{ odd}). \end{cases}$

Also $1_B^G \cong S_G \oplus S_{G(h-1)}$.

These results are immediate from §4.4; the first result is well-known ([16],Thm.3.3).

For the rest of this section we shall consider the case when $n=3$. Recall from chapter 3 that we have defined representations $I(i,j)$ for $0<i,j<h$, $i+j \geq h$, and $R(i,j)$ for $2<i,j<h$, $i+j > h+1$. We extend our notation by defining $I(0,h-1)$, $I(h-1,0)$ to be the permutation representations of G on the sets $\mathfrak{F}(0,h-1)$, $\mathfrak{F}(h-1,0)$ of all submodules of V of types $(h,h,1)$, $(h-1)$ respectively ; and put $I(0,0) = 1$.

We also define

$$R(1,h) = I(1,h) - I(0,h) - I(1,h-1) + I(0,h-1) \;,$$

$$R(h,1) = I(h,1) - I(h,0) - I(h-1,1) + I(h-1,0) \;,$$

$$R(0,h) = I(0,h) - I(0,h-1) \;,\; R(h,0) = I(h,0) - I(h-1,0).$$

Finally, put $Z = I(1,1) - 1_{B(h-1)}^{G(h-1)}$.

The following is well-known.

<u>Proposition 5.32</u>

Assume that $h=1$. Then

(i) $St_G = R(1,1) = 1_B^G - I(1,0) - I(0,1) + I(0,0)$ is irreducible of dimension q^3.

(ii) $R(0,1)$ and $R(1,0)$ are isomorphic, irreducible and of dimension $q(q+1)$.

(iii) $1_B^G \cong St_G + 2.R(1,0) + 1$ is a decomposition into distinct irreducible components (with multiplicities).

In case $h \geq 2$ we have the following result.

<u>Proposition 5.33</u>

(i) For $0<i<h-2$ $R(h-i,h)$ and $R(h,h-i)$ are irreducible of dimension $(q^2-1)(q^3-1)q^{3h-5-i}$.

(ii) $R(1,h)$ and $R(h,1)$ are irreducible of dimension $(q^2-1)(q^3-1)q^{2h-4}$.

(iii) $R(0,h)$ and $R(h,0)$ are isomorphic, irreducible and of dimension $(q+1)(q^3-1)q^{2h-4}$.

It is likely that if $1 \leq i \leq h-1$ then $R(h-i,h)$ and $R(h,h-i)$ are not isomorphic; this statement is motivated by the fact that $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi^i \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \pi^i & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ are not conjugate (§5.21). We shall show that this is true if $h=2$ and $i=1$.

Proposition 5.34

Let $h=2$. Then $R(1,2)$ and $R(2,1)$ are not isomorphic, $Z \cong R(2,0) \cong R(0,2)$ (so Z is irreducible) and the following are decompositions into irreducible components.

$1_B^G \cong St_G + R(1,2) + R(2,1) + 3.R(2,0) + St_{G(1)} + 2.R(1,0) + 1$

$S_G \cong St_G + R(2,0) + 1.$

Remarks

§5.34 and §5.22 show that in case $n=3, h=2$ the number of distinct irreducible components of 1_B^G equals the number of unipotent conjugacy classes of G; there exists an apparently 'natural' bijection in which St_G corresponds to the regular unipotent class and 1 to the identity, as follows.

class representative	:	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \pi & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
representation	:	St_G	$R(2,1)$	$R(1,2)$	$R(2,0)$

class representative	:	$\begin{pmatrix} 1 & \pi & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \pi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
representation	:	$St_{G(1)}$	$R(1,0)$	1

In case $h=1$ such 'natural' bijections are well-known, since unipotent conjugacy classes and components of 1_B^G can both be parametrised by partitions of n. In case $n=2$ and $h \geq 2$ there is also a bijection : $\begin{pmatrix} 1 & \pi^i \\ 0 & 1 \end{pmatrix} \longleftrightarrow St_{G(i)}$ $(0<i<h)$.

Proof of §5.33

(i) see §3.24/6.

For (ii) and (iii) we need first to observe that (in a notation to be explained below) by §3.1 we have

$$\begin{tikzcd}
 & \mathfrak{F}(0,h) \arrow[dl, shift right, "q+1"'] \arrow[dr, shift left, "1"] & \\
\mathfrak{F}(1,h) \arrow[ur, shift right, "1"'] \arrow[dr, shift left, "1"] & & \mathfrak{F}(0,h-1) \arrow[ul, shift left, "q^2"] \arrow[dl, shift left, "q(q+1)"] \\
 & \mathfrak{F}(1,h-1) \arrow[ul, shift left, "q"] \arrow[ur, shift left, "1"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathfrak{F}(h,0) \arrow[dl, shift right, "q+1"'] \arrow[dr, shift left, "1"] & \\
\mathfrak{F}(h,1) \arrow[ur, shift right, "1"'] \arrow[dr, shift left, "1"] & & \mathfrak{F}(h-1,0) \arrow[ul, shift left, "q^2"] \arrow[dl, shift left, "q(q+1)"] \\
 & \mathfrak{F}(h-1,1) \arrow[ul, shift left, "q"] \arrow[ur, shift left, "1"] &
\end{tikzcd}$$

Each arrow above, for example $\mathfrak{F}(1,h) \xleftarrow{q+1} \mathfrak{F}(0,h)$ indicates a statement of the form : given $\mathfrak{F} \in \mathfrak{F}(0,h)$ there exist precisely q+1 flags $\mathfrak{G} \in \mathfrak{F}(1,h)$ such that $\mathfrak{G} \geq \mathfrak{F}$. Similarly, $\mathfrak{F}(h.0) \xrightarrow{1} \mathfrak{F}(h-1,0)$ asserts that given $\mathfrak{F} = (0 \subset V_1 \subset V)$ in $\mathfrak{F}(h,0)$ (so $V_1 \sim (h)$) there exists a unique $U \in \mathfrak{F}(h-1,0)$ such that $U \subset V_1$. All these statements follow from §3.12/3/4.

It follows that the representations R(1,h), R(h,1), R(0,h), and R(h,0) may each be constructed by the procedure of §3.2 as the sum of the reduced homologies of $|\mathfrak{F}(0,h-1)|$ copies of a suitable complex X. In the latter two cases X is a discrete set of q^2 points. In the former two cases X is of dimension 1 and is path-connected, since, for example, if $U \in \mathfrak{F}(h-1,0)$, $V \in \mathfrak{F}(h,0)$ with $\pi V = U$, and $(U \subset W) \in \mathfrak{F}(h-1,1)$ then either $V \subset W$, in which case the 1-simplex $(V \subset W) \in \mathfrak{F}(h,1)$ connects V to $(U \subset W)$,

or $V \not\subset W$, in which case we choose $V' \sim (h)$ with $U \subset V' \subset W$; then $W' = \langle V, V' \rangle \sim (h,1)$ and $(V' \subset W)$, $(V' \subset W')$, $(V \subset W')$ connect $(U \subset W)$ to V via V' and $(U \subset W')$.

So in all cases the corresponding augmented chain complex is exact. Moreover $R(1,h)$ is the subrepresentation of $I(1,h)$ defined by certain cycle conditions (compare §3.23(ii)); and similarly for $R(h,1)$, $R(0,h)$ and $R(h,0)$.

We can now compute the dimensions of these representations.

$$\begin{aligned}\dim(R(1,h)) = \dim(R(h,1)) &= |\mathcal{F}(0,h-1)|.(q^2(q+1)-q(q+1)-q^2+1)\\ &= (q^2+q+1)q^{2h-4}(q^2-1)(q-1) \text{ by §3.12.}\\ &= (q^3-1)(q^2-1)q^{2h-4}.\\ \dim(R(0,h)) = \dim(R(h,0)) &= |\mathcal{F}(h-1,0)|.(q^2-1)\\ &= (q^2+q+1)q^{2h-4}(q^2-1)\\ &= (q+1)(q^3-1)q^{2h-4}.\end{aligned}$$

To prove $R(h,1)$ irreducible we shall prove that $< I(h,1),R(h,1) >_G = 1$, using the cycle conditions, as in §3.25. So let $\mathcal{F}_0 = (0 \subset V_1 \subset V_2 \subset V)$ be the standard flag of type $(h,1)$, i.e. $V_1 = < e_1 >$, $V_2 = < e_1,\pi^{h-1}e_2 >$. Then

$$B_{(h,1)} = \mathrm{Stab}_G \mathcal{F}_0 = \left\{ \begin{pmatrix} a & b & c\\ 0 & e & f\\ 0 & j & k\end{pmatrix} : a,e,k \in R^*; f,b,c \in R; j \in \pi R \right\}.$$

Now §3.17/8 hold for $\underset{\sim}{a} = (h,1)$ so as in §3.25

$$X = \{vw : w \in W,\ v \in U^-(\pi R) \cap {}^wU^-(\pi R)\}$$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0\\ a & 1 & 0\\ b & c & 1\end{pmatrix}, \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ c & b & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ a & 0 & 1\\ b & 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ c & 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 & 0\\ 0 & a & 1\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} : a,b,c \in \pi R \right\}$$

forms a sufficient set of representatives for $B_{(h,1)}-B_{(h,1)}$ double cosets in G. We could proceed with the proof by the method given in §3.25, but we may alternatively proceed as follows. Writing $a = \alpha\pi^r$, $b = \beta\pi^s$, $c = \gamma\pi^t$ with $\alpha,\beta,\gamma \in R^*$ the equations

$$\begin{pmatrix} 1 & 0 & 0\\ a & 1 & 0\\ b & c & 1\end{pmatrix} = \begin{pmatrix} \beta^{-1} & 0 & 0\\ 0 & \alpha\beta^{-1} & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix} 1 & 0 & 0\\ \pi^r & 1 & 0\\ \pi^s & \pi^t & 1\end{pmatrix} \begin{pmatrix} \beta & 0 & 0\\ 0 & \beta\alpha^{-1} & 0\\ 0 & (\gamma-\beta\alpha^{-1})\pi^t & 1\end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ c & b & 1\end{pmatrix} = \begin{pmatrix} \beta^{-1} & 0 & 0\\ 0 & \gamma^{-1} & 0\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ \pi^t & \pi^s & 1\end{pmatrix} \begin{pmatrix} \gamma & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\a&0&1\\b&1&0\end{pmatrix}=\begin{pmatrix}\beta^{-1}&0&0\\0&\alpha\beta^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\\pi^r&0&1\\\pi^s&1&0\end{pmatrix}\begin{pmatrix}\beta&0&0\\0&1&0\\0&0&\beta\alpha^{-1}\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\c&1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\gamma^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\\pi^t&1&0\end{pmatrix}\begin{pmatrix}\gamma&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&a&1\\1&0&0\end{pmatrix}=\begin{pmatrix}\alpha^{-1}&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&\pi^r&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&\alpha&0\\0&0&1\end{pmatrix}$$

show that in fact the subset of X with $a=\pi^r$, $b=\pi^s$, $c=\pi^t$ forms a sufficient set of representatives of $B_{(h,1)}-B_{(h,1)}$ cosets in G and it follows that the $B_{(h,1)}$-orbit type of a flag $\mathfrak{F}=(0\subset U_1\subset U_2\subset V)\in\mathcal{F}(h,1)$ is determined by the isomorphism types (a_{ij}) of $U_i\cap V_j$ $(1\leq i,j\leq 2)$.

Now let $f:\mathcal{F}(h,1)\to\mathbb{C}$ be constant on $B_{(h,1)}$-orbits and satisfy the cycle conditions

C_1 : $\sum_{\mathfrak{F}\geq\mathfrak{G}} f(\mathfrak{F})=0$ for each $\mathfrak{G}=(W_1\subset U_2)\in\mathcal{F}(h-1,1)$

C_2 : $\sum_{\mathfrak{F}\geq\mathfrak{H}} f(\mathfrak{F})=0$ for each $\mathfrak{H}=(U_1)\in\mathcal{F}(h,0)$.

As in the proof of §3.25 we must show that either $f(\mathfrak{F})=0$ or there exists $k\in\mathbb{Q}^*$ such that $f(\mathfrak{F})=k.f(\mathfrak{F}_0)$. ------(*)

We first apply condition C_1. So let $\mathfrak{G}=(W_1\subset U_2)\in\mathcal{F}(h-1,1)$ and consider the q flags $\mathfrak{F}=(U_1\subset U_2)\in\mathcal{F}(h,1)$ with $\mathfrak{F}\geq\mathfrak{G}$. U_2 is the same for all these flags $\mathfrak{F}$ and so $U_2\cap V_j$ has the same type for all $\mathfrak{F}$ $(j=1,2)$.

<u>Case 1</u> : $W_1\not\subset V_1$.

$U_1\cap V_1$ is the unique submodule of U_1 of type (i) for some i; now if i=h then $W_1\subset U_1=U_1\cap V_1=V_1$, and if i=h-1 then $W_1=U_1\cap V_1\subset V_1$, so we must have $i\leq h-2$. So $U_1\cap V_1$ is the unique submodule of W_1 of its type. Hence $U_1\cap V_1=W_1\cap V_1$ for all q flags $\mathfrak{F}\geq\mathfrak{G}$.

But also $U_1 \cap V_2 \subset W_1$, since if this were not so then $U_1 \subset V_2$ whence $U_1 \cap V_1$ has type (h) or $(h-1)$ (because $|V_2|/|V_1| = q$), a contradiction. Hence $U_1 \cap V_2 = W_1 \cap V_2$ for all q flags $\mathfrak{F}$. Hence (a_{ij}) is the same for all the $\mathfrak{F}$ and so by condition C_1, f being constant on $B_{(h,1)}$-orbits, we have $f(\mathfrak{F}) = 0$ (for any $\mathfrak{F} = (U_1 \subset U_2)$ with $\pi U_1 \not\subset V_1$).

<u>Case 2</u> : $W_1 \subset V_1$.

(a) $U_2 \cap V_2 \sim (h,1)$. Then $U_2 = V_2$ so $U_1 \cap V_2 = U_1$: this has the same type for all the $\mathfrak{F}$. One $\mathfrak{F}$ has $U_1 \cap V_1 = V_1$ $(U_1 = V_1)$ and the remaining $(q-1)\,\mathfrak{F}$ have $U_1 \cap V_1 = W_1$.

$$\text{Hence} \quad f\begin{pmatrix} h & h \\ h & h,1 \end{pmatrix} + (q-1).f\begin{pmatrix} h-1 & h \\ h & h,1 \end{pmatrix} = 0 \quad \text{——}(1),$$

where we write only the orbit type (a_{ij}) since f is constant on $B_{(h,1)}$-orbits.

(b) $U_2 \cap V_2 \sim (h), V_1 \subset U_2$. Then $U_2 \cap V_1 \sim (h)$, so $U_2 \cap V_2 = V_1$. All but one $\mathfrak{F}$ have $U_1 \cap V_1 = W_1$ and so also $U_1 \cap V_2 \subset U_2 \cap V_2 = V_1$, whence $U_1 \cap V_2 = U_1 \cap V_1 = W_1$.

$$\text{Hence} \quad f\begin{pmatrix} h & h \\ h & h \end{pmatrix} + (q-1).f\begin{pmatrix} h-1 & h-1 \\ h & h \end{pmatrix} = 0 \quad \text{——}(2).$$

(c) $U_2 \cap V_2 \sim (h)$, $V_1 \not\subset U_2$. Then $V_1 \supsetneq U_2 \cap V_1 \supset W_1$, so $U_2 \cap V_1 = W_1$. Also $U_1 \cap V_1 = W_1$ for all $\mathfrak{F}$; one $\mathfrak{F}$ has $U_1 \cap V_2 = U_1$ $(U_1 \subset V_2)$ and the other $(q-1)\,\mathfrak{F}$ have $U_1 \cap V_2 = W_1$.

$$\text{Hence} \quad f\begin{pmatrix} h-1 & h \\ h-1 & h \end{pmatrix} + (q-1).f\begin{pmatrix} h-1 & h-1 \\ h-1 & h \end{pmatrix} = 0 \quad \text{——}(3).$$

(d) $U_2 \cap V_2 \sim (i)$, $i < h-1$. Now $W_1 \subset U_1 \cap V_1 \subset U_2 \cap V_2$ so $i = h-1$, and $W_1 = U_1 \cap V_1 = U_1 \cap V_2 = U_2 \cap V_1 = U_2 \cap V_2$, the same for all $\mathfrak{F}$; so f vanishes here.

(e) $U_2 \cap V_2 \sim (i,1)$, $i < h-1$. Now $W_1 \subset U_1 \cap V_1 \subset U_2 \cap V_2$ so $i = h-1$. Then $U_1 \cap V_1 = W_1 = U_1 \cap V_2$, the same type for all $\mathfrak{F}$; so f vanishes here too.

We now note that all orbit types have occured; we relate equations (1), (2) and (3) by applying condition C_2.

So let $\mathcal{K} = (U_1) \in \mathcal{F}(h,0)$ and consider the q+1 flags $\mathcal{J} = (U_1 \subset U_2) \in \mathcal{F}(h,1)$ with $\mathcal{J} \geq \mathcal{K}$.

<u>Case 1</u> : $U_1 = V_1$.

Then $U_1 \cap V_2 = U_1$, $U_2 \cap V_1 = V_1$. One $\mathcal{J}$ has $U_2 = V_2$, so $U_2 \cap V_2 \sim (h,1)$; the other q $\mathcal{J}$ have $U_2 \cap V_2 \sim (h)$.

$$\text{Hence} \quad f\begin{pmatrix} h & h \\ h & h,1 \end{pmatrix} + q.f\begin{pmatrix} h & h \\ h & h \end{pmatrix} = 0 \qquad \text{——(4)}$$

<u>Case 2</u> : $U_1 \cap V_1 \sim (h-1)$, $U_1 \subset V_2$.

Then one $\mathcal{J}$ has $U_2 \supset V_1$, so $U_2 \cap V_1 \sim (h)$, and $U_2 \cap V_2 \sim (h,1)$ (it can't be type (h) since if it were then $U_1 = V_1$, which contradicts $U_1 \cap V_1 \sim (h-1)$). The other q $\mathcal{J}$ have $U_2 \cap V_1 = U_1 \cap V_1 \sim (h-1)$ and $U_2 \cap V_2 = U_1 \sim (h)$.

$$\text{Hence} \quad f\begin{pmatrix} h-1 & h \\ h & h,1 \end{pmatrix} + q.f\begin{pmatrix} h-1 & h \\ h-1 & h \end{pmatrix} = 0 \qquad \text{——(5).}$$

Equations (1) to (5) are now sufficient to prove (*) and hence the irreducibility of $R(h,1)$.

It follows that $R(1,h)$ is also irreducible since it is constructed in a manner symmetrical to the construction of $R(h,1)$ from the geometry of $X(V)$. (see §4.4 — in particular $R(1,h)(g) = R(h,1)(\theta(g))$ for $g \in G$, where $\theta(g)$ is the reflection of the matrix g in the secondary diagonal, as in §3.5).

We employ the same method to prove that $R(h,0)$ is irreducible; in fact we show $\langle I(h,0), R(h,0) \rangle_G = 1$.

So fix a flag $\mathcal{J}_0 = (V_1) \in \mathcal{F}(h,0)$ and let $P = \mathrm{Stab}_G(V_1)$. Then $P \supset B_{(h,1)}$ so the P-orbit of a flag $\mathcal{J} = (U) \in \mathcal{F}(h,0)$ is determined by the isomorphism type of $U \cap V_1$.

If $f \in R(h,0)$ then it satisfies the cycle condition

C_3 : $\sum_{U \supset W} f(U) = 0$, for each $W \sim (h-1)$ (sum over the q^2 flags $\mathcal{J} = (U) \in \mathcal{F}(h,0)$ with $U \supset W$.).

Assume that f is constant on P-orbits.

<u>Case 1</u> : $W \subset V_1$.

Then one of the q^2 $\mathfrak{Y}$ has $U = V_1$; the other q^2-1 $\mathfrak{Y}$ have $U \cap V_1 = W$. Hence $f(h) + (q^2-1).f(h-1) = 0$ ----(**).

(Again just writing the orbit types).

<u>Case 2</u> : $W \not\subset V_1$.

Then $U \cap V_1$ is the unique submodule of V_1 of type (i) for some i; now if i=h then $W \subset U = U \cap V_1 = V_1$, and if i=h-1 then $W = U \cap V_1 \subset V_1$, so we must have $i \leq h-2$ and so $U \cap V_1$ is the unique submodule of its type $\subset W$. Hence $U \cap V_1$ is the same for all q^2 $\mathfrak{Y}$, and so all q^2 $\mathfrak{Y}$ lie in the same orbit.

Hence f vanishes here, and this together with (**) is sufficient to prove R(h,0) irreducible.

(Exactly this type of proof can be used to prove St_G is irreducible in case $G = GL_2(R)$.)

Finally, $I(h,0) \cong 1_{P_{\{1\}}}^{G} \cong 1_{P_{\{2\}}}^{G} \cong I(0,h)$ and $I(h-1,0) \cong 1_{P_{\{1\}}(h-1)}^{G(h-1)} \cong 1_{P_{\{2\}}(h-1)}^{G(h-1)} \cong I(0,h-1)$ and so $R(h,0) \cong R(0,h)$.

The proof of §5.33 is complete. ∎

<u>Proof of §5.34</u>

Let $g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix}$ and $h = \theta(g) = \begin{pmatrix} 1 & \pi & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. We shall show that $R(2,1)(g) \neq R(2,1)(h)$; since the latter equals $R(1,2)(g)$ it follows that $R(2,1)$ cannot be isomorphic to $R(1,2)$.

Now $I(1,1)(g) = I(1,1)(\pi_{h-1}(g)) = I(1,1)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I(1,1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I(1,1)(h)$, and similarly $I(1,0)(g) = I(1,0)(h)$. So $R(2,1)(g) - R(2,1)(h) = I(2,1)(g) - I(2,0)(g) - I(2,1)(h) + I(2,0)(h)$. We shall compute this directly : note that, for example, $I(2,0)(g) = |\{\text{flags of type } (2,0) \text{ which are fixed by } g\}|$.

Now the flags of type (2,0) are $\begin{cases} < e_1+re_2+se_3 > \\ < le_1+e_2+se_3 > \\ < le_1+me_2+e_3 > \end{cases}$ $(r,s \in R,\ l,m \in \pi R)$.

$< e_1+re_2+se_3 >$ is fixed by g iff $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ r \\ s \end{pmatrix} = k \begin{pmatrix} 1 \\ r \\ s \end{pmatrix}$ for

some $k \in R$ iff $k=1+r,\ kr=r+\pi s,\ ks=s$

iff $r^2=\pi s,\ rs=0$

iff $\pi|r,\ \pi|s$: there are q^2 solutions.

$< le_1+e_2+se_3 >$ is fixed by g iff $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ 1 \\ s \end{pmatrix} = k \begin{pmatrix} l \\ 1 \\ s \end{pmatrix}$ for

some $k \in R$ iff $kl=l+1,\ k=1+\pi s,\ ks=s$

iff $1=l\pi s,\ \pi s^2=0$: there are no solutions.

$< le_1+me_2+e_3 >$ is fixed by g iff $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ m \\ 1 \end{pmatrix} = k \begin{pmatrix} l \\ m \\ 1 \end{pmatrix}$ for

some $k \in R$ iff $kl=l+m,\ km=m+\pi,\ k=1$

iff $m=0,\ 0=\pi$: there are no solutions.

Hence $I(2,0)(g) = q^2$.

To compute $I(2,1)(g)$ we only need consider the flags $\mathfrak{F} = (U_1 \subset U_2)$ of type (2,1) with $U_1 = < e_1+re_2+se_3 >$ $(\pi|r,s)$ and $U_2 = < U_1, \pi e_2+le_3 >$ $(\pi|l)$ or $< U_1, \pi e_3 >$.

$< U_1, \pi e_2+le_3 >$ is fixed by g iff $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pi \\ l \end{pmatrix} = k \begin{pmatrix} 0 \\ \pi \\ l \end{pmatrix} + j \begin{pmatrix} 1 \\ r \\ s \end{pmatrix}$

for some $k,j \in R$ iff $j=\pi,\ k\pi+jr=\pi+\pi l,\ kl+js=l$

iff $k\pi=\pi,\ kl=l$: q^3 solutions.

$< U_1, \pi e_3 >$ is fixed by g iff $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \pi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix} = k \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix} + j \begin{pmatrix} 1 \\ r \\ s \end{pmatrix}$

for some $k,j \in R$ iff $j=0,\ jr=\pi^2,\ k\pi+js=\pi$

iff $k\pi=\pi$: q^2 solutions.

Hence $I(2,1)(g) = q^3 + q^2$.

$< e_1+re_2+se_3 >$ is fixed by h iff $\begin{pmatrix} 1 & \pi & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ r \\ s \end{pmatrix} = k \begin{pmatrix} 1 \\ r \\ s \end{pmatrix}$ for

some $k \in R$ iff $k=1+\pi r,\ kr=r+s,\ ks=s$

iff $\pi r^2=s,\ \pi rs=0$: q^2 solutions.

$< le_1+e_2+se_3 >$ is fixed by h iff $\begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}l\\1\\s\end{pmatrix} = k\begin{pmatrix}l\\1\\s\end{pmatrix}$ for

some $k \in R$ iff $kl=l+\pi$, $k=1+s$, $ks=s$

iff $ls=\pi$, $s^2=0$

iff $\pi|s$, $0=ls=\pi$: no solutions.

$< le_1+me_2+e_3 >$ is fixed by h iff $\begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}l\\m\\1\end{pmatrix} = k\begin{pmatrix}l\\m\\1\end{pmatrix}$ for

some $k \in R$ iff $kl=l+\pi m$, $km=m+1$, $k=1$

iff $0=\pi m$, $0=1$: no solutions.

Hence $I(2,0)(h) = q^2$.

To compute $I(2,1)(h)$ we only need consider the flags $\mathcal{F} = (U_1 \subset U_2)$ of type $(2,1)$ with $U_1 = < e_1+re_2+\pi r^2e_3 >$ and $U_2 = < U_1, \pi e_2+le_3 >$ or $< U_1, \pi e_3 >$ $(\pi|l)$.

$< U_1, \pi e_2+le_3 >$ is fixed by h iff $\begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0\\\pi\\l\end{pmatrix} = k\begin{pmatrix}0\\\pi\\l\end{pmatrix} + j\begin{pmatrix}1\\r\\\pi r^2\end{pmatrix}$

for some $k,j \in R$ iff $j=\pi^2$, $k\pi+jr=\pi+l$, $kl+j\pi r^2=l$

iff $(k-1)\pi=l$, $(k-1)l=0$

iff $(k-1)^2\pi=0$, i.e. $\pi|(k-1)$ so $l=0$:

there are q^2 solutions.

$< U_1, \pi e_3 >$ is fixed by h iff $\begin{pmatrix}1&\pi&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0\\0\\\pi\end{pmatrix} = k\begin{pmatrix}0\\0\\\pi\end{pmatrix} + j\begin{pmatrix}1\\r\\\pi r^2\end{pmatrix}$

for some $k,j \in R$ iff $j=0$, $jr=\pi$, $k\pi+j\pi r^2=\pi$

iff $0=\pi$: no solutions.

Hence $I(2,1)(h) = q^2$.

Hence $R(2,1)(g) - R(2,1)(h) = (q^3+q^2) - q^2 - q^2 + q^2 \neq 0$, so $R(2,1) \ncong R(1,2)$.

To prove $Z \cong R(2,0)$ we must show that

$$I(2,0) + 1_{B(1)}^{G(1)} \cong I(1,1) + I(1,0) \qquad ----(*).$$

So fix a submodule U of V of type (1); this corresponds to a certain vertex in $X(V)$. Recall $\mathrm{link}(U) \cong T(\overline{V})$ (§4.15)

Now {vertices of type (2) in link(U)} ∪ {vertices of type (1,1) in link(U)}

= {vertices in link(U) whose quotient by U has type (1)}

≈ {lines in $\overline{V}$}.

Also {vertices of type (2,1) in link(U)} ∪ {vertex (0)}

= {vertices in link(U) whose quotient by U (or vice versa) has type (1,1)}

≈ {planes in $\overline{V}$}.

The usual duality between lines and planes in $\overline{V}$ yields an isomorphism of the two permutation representations of $\mathrm{Stab}_G U$ on these sets; and then summing over all U of type (1) we obtain the isomorphism (*) of representations of G.

Finally, from the definitions we have

$$1_B^G = St_G+R(2,1)+R(2,0)+R(1,2)+R(0,2)+Z+St_{G(1)}+R(1,0)+R(0,1)+1$$
$$\cong St_G+R(2,1)+R(1,2)+3.R(2,0)+St_{G(1)}+2.R(1,0)+1 \ .$$

Also

$$\begin{aligned} St_G+R(2,0)+1 &\cong St_G+R(2,0)+R(0,2)-Z+1 \\ &= I(2,2)-(I(2,1)-I(2,0))-(I(1,2)-I(0,2)) \\ &\quad +(1_{B(1)}^{G(1)}-I(1,0)-I(0,1)+1) \\ &= S_G \quad \text{by §4.4 and §5.32.} \end{aligned}$$

This completes the proof of §5.34. ∎

5.4 A counterexample on the character of S_G

In case h=1 the character of S_G is always 0 or ± a power of q. We show that this is not always the case if h≠1.

Proposition 5.41

Let n=3,h=3. Let $g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$. Then $S_G(g) = q^3(q-1)^2$.

Proof:

We use the formula $S_G = I(h,h) + \sum_{i=0}^{h-1} (-1)^{h-i}(I(h,i)+I(i,h)) + S_{G(h-1)}$.

The calculation is similar to that in §5.34. Observe that g is conjugate to $\theta(g) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ so $I(h,i)(g) = I(i,h)(g)$.

Case i=0

The flags of type (h,0) are $\begin{cases} < e_1+re_2+se_3 > \\ < le_1+e_2+se_3 > \\ < le_1+me_2+e_3 > \end{cases}$ $(r,s \in R,\ l,m \in \pi R)$.

$< e_1+re_2+se_3 >$ is fixed by g iff $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1\\r\\s\end{pmatrix} = k\begin{pmatrix}1\\r\\s\end{pmatrix}$ for

some $k \in R$ iff $k=1+r,\ kr=r,\ ks=s$

iff $r^2=0,\ rs=0$.

$< le_1+e_2+se_3 >$ is fixed by g iff $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}l\\1\\s\end{pmatrix} = k\begin{pmatrix}l\\1\\s\end{pmatrix}$ for

some $k \in R$ iff $kl=l+1,\ k=1,\ ks=s$

iff $l=l+1$: there are no solutions.

$< le_1+me_2+e_3 >$ is fixed by g iff $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}l\\m\\1\end{pmatrix} = k\begin{pmatrix}l\\m\\1\end{pmatrix}$ for

some $k \in R$ iff $kl=l+m,\ km=m,\ k=1$

iff $m=0$.

So the flags of type (h,0) fixed by g are

$< e_1+re_2+se_3 >$ $(r^2=0=rs)$, $< le_1+e_3 >$ $(\pi|l)$.

If h=2 then $\pi|r$; if $\pi\|r$ then $\pi|s$. So the number of these fixed flags is $(q-1)q+q^2+q = 2q^2$. $(=I(2,0)(g))$.

If h=3 then $\pi^2|r$; if $\pi^2\|r$ then $\pi|s$. So the number of these fixed flags is $(q-1)q^2+q^3+q^2 = 2q^3$. $(=I(3,0)(g))$.

Case i > 0

The flags of type (h,i) to be considered are $(V_1 \subset V_2)$ where

either (a) $V_1 = < e_1+re_2+se_3 >$ $(r^2=0=rs)$, $V_2 = < V_1, \pi^{h-i}e_2+le_3 >$ $(\pi^{h-i}|l)$ or $V_2 = < V_1, \pi^{h-i}e_3+me_2 >$ $(\pi^{h-i+1}|m)$.

or (b) $V_1 = < te_1+e_3 >$ $(\pi|t)$, $V_2 = < V_1, \pi^{h-i}e_1+le_2 >$ $(\pi^{h-i}|l)$ or $V_2 = < V_1, \pi^{h-i}e_2+me_1 >$ $(\pi^{h-i+1}|m)$.

(a)(i) $< V_1, \pi^{h-i}e_2+le_3 >$ is fixed by g iff there are $k,j \in R$ with $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\\pi^{h-i}\\l\end{pmatrix} = k\begin{pmatrix}1\\r\\s\end{pmatrix} + j\begin{pmatrix}0\\\pi^{h-i}\\l\end{pmatrix}$

iff $k=\pi^{h-i},\ kr+j\pi^{h-i}=\pi^{h-i},\ ks+jl=l$

iff $\pi^{h-i}(r+j-1)=0,\ (j-1)l=-\pi^{h-i}s$.

(a)(ii) $< V_1, \pi^{h-i}e_3+me_2 >$ is fixed by g iff there are $k,j \in R$ with $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\m\\\pi^{h-i}\end{pmatrix} = k\begin{pmatrix}1\\r\\s\end{pmatrix} + j\begin{pmatrix}0\\m\\\pi^{h-i}\end{pmatrix}$

iff $k=m$, $kr+jm=m$, $ks+j\pi^{h-i}=\pi^{h-i}$

iff $m(r+j-1)=0$, $(j-1)\pi^{h-i}=-ms$.

(b)(i) $< V_1, \pi^{h-i}e_1+le_2 >$ is fixed by g iff there are $k,j \in R$ with $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\pi^{h-i}\\l\\0\end{pmatrix} = k\begin{pmatrix}t\\0\\1\end{pmatrix} + j\begin{pmatrix}\pi^{h-i}\\l\\0\end{pmatrix}$

iff $kt+j\pi^{h-i}=\pi^{h-i}+l$, $jl=l$, $k=0$

iff $(j-1)l=0$, $(j-1)\pi^{h-i}=l$.

(b)(ii) $< V_1, \pi^{h-i}e_2+me_1 >$ is fixed by g iff there are $k,j \in R$ with $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}m\\\pi^{h-i}\\0\end{pmatrix} = k\begin{pmatrix}t\\0\\1\end{pmatrix} + j\begin{pmatrix}m\\\pi^{h-i}\\0\end{pmatrix}$

iff $kt+jm=m+\pi^{h-i}$, $j\pi^{h-i}=\pi^{h-i}$, $k=0$

iff $(j-1)m=\pi^{h-i}$, $(j-1)\pi^{h-i}=0$.

Note that in case (b)(ii) there are no solutions, since in the first equation $\pi^{h-i+1}\,|\,(j-1)m$ whereas $(j-1)m=\pi^{h-i} \neq 0$ $(i > 0)$. We compute the number N of solutions for each of the other cases.

<u>h=2</u> : <u>i=1</u> :

(a)(i) : $\pi(r+j-1)=0$ and $\pi|r$ so $\pi(j-1)=0$. So $\pi|(j-1)$ and so $-\pi s=(j-1)l=0$. So $\pi|s$. Hence $N = q^3$ in this case.

(a)(ii) : $m=0$. So $N = 2q^2-q$.

(b)(i) : $\pi|l$ and $(j-1)l=0$. So if $\pi\|l$ then $\pi|(j-1)$ and so $l=(j-1)\pi=0$, contradiction. So $l=0$. Hence $N = q$.

Hence $I(2,1)(g) = q^3+2q^2$.

<u>h=2</u> : <u>i=2</u> :

(a)(i) : $j-1=-r$ so $s=rl$. Thus $rs=r^2l=0$ for any l. So $N = q^3$.

(a)(ii) : $\pi|m$ so $(j-1)m=0$ and $j-1=-ms$. So $N = (2q^2-q).q$

(b)(i) : $j-1=l$ so $l^2=0$, i.e. $\pi|l$. Hence $N = q^2$.

Hence $I(2,2)(g) = 3q^3$.

<u>h=3</u> : <u>i=1</u> :

(a)(i) : $\pi^2(r+j-1)=0$ and $\pi^2|r$ so $\pi^2(j-1)=0$. So $\pi|(j-1)$ and so $-\pi^2 s=(j-1)l=0$. So $\pi|s$. Hence $N = q^4$.

(a)(ii) : $m=0$. So $N = 2q^3-q^2$.

(b)(i) : $\pi^2|l$ and $(j-1)l=0$. So if $\pi^2\nmid l$ then $\pi|(j-1)$ and so $l=(j-1)\pi^2=0$, contradiction. So $l=0$. Hence $N = q^2$.

Hence $I(3,1)(g) = q^4+2q^3$.

<u>h=3</u> : <u>i=2</u> :

(a)(i) : $\pi(r+j-1)=0$ and $\pi^2|r$ so $\pi(j-1)=0$. So $\pi^2|(j-1)$ and so $-\pi s=(j-1)l=0$. So $\pi^2|s$. Hence $N = q^4$.

(a)(ii) : $\pi^2|m$ so $(j-1)m=0$ and $(j-1)\pi=-ms$. Hence $\pi|(j-1)$ and no other restrictions. So $N = q.(2q^3-q^2)$.

(b)(i) : $\pi|l$ and $(j-1)l=0$. $l=(j-1)\pi$ so $(j-1)^2\pi=0$. So $\pi|(j-1)$ and so $\pi^2|l$. Hence $N = q^3$.

Hence $I(3,2)(g) = 3q^4$.

<u>h=3</u> : <u>i=3</u> :

(a)(i) : $j-1=-r$ so $s=rl$. Thus $rs=r^2l=0$ for any l. So $N = q^4$.

(a)(ii) : $\pi|m$ so $(j-1)m=0$. $j-1=-ms$ so $m^2s=0$. If $\pi\nmid m$ then $\pi|s$ and $rs=0$ always. If $\pi^2|m$ then $m^2s=0$ always. Hence $N = (q^2-q).q^2.q + q.(2q^3-q^2) = q^5+q^4-q^3$.

(b)(i) : $j-1=l$ so $l^2=0$ and so $\pi^2|l$. Hence $N = q^3$.

Hence $I(3,3)(g) = q^5+2q^4$.

Now by §2.22 $S_{G(1)}(g) = 0$. So by the above calculations

$$\begin{aligned} S_{G(2)}(g) &= I(2,2)(g) - 2(I(2,1)(g)-I(2,0)(g)) + S_{G(1)}(g) \\ &= 3q^3 - 2(q^3+2q^2-2q^2) + 0 \\ &= q^3. \end{aligned}$$

$$\begin{aligned} S_G(g) &= I(3,3)(g) - 2.(I(3,2)(g)-I(3,1)(g)+I(3,0)(g)) + S_{G(2)}(g) \\ &= q^5+2q^4 -2(3q^4-q^4-2q^3+2q^3) + q^3 \\ &= q^5-2q^4+q^3 \quad = \quad q^3(q-1)^2. \end{aligned}$$

∎

5.5 The character of St_G at split semisimple elements.

Proposition 5.51

Let $t \in G$ be split semisimple and assume that $h \geq 2$.

(i) Let $n=2$ and write $k = k_{12}(t)$. Then

$$St_G(t) = \begin{cases} 0 & k \leq h-2 \\ q^{h-2}(q-1) & k = h-1 \\ q^{h-2}(q^2-1) & k = h \end{cases}$$

(ii) Let $n=3$. Then there exists $w \in W$ such that

$$k = k_{12}({}^wt) \geq k_{23}({}^wt) = k_{13}({}^wt) = l, \text{ say, and}$$

$$St_G(t) = \begin{cases} 0 & l \leq h-2 \\ q^{3h-5}(q-1)^2 & k = h-1 = l \\ q^{3h-5}(q^2-1)(q-1) & k = h,\ l = h-1 \\ q^{3h-5}(q^2-1)(q^3-1) & k = h = l \end{cases}$$

These results lead us to make the following conjecture for the general case $n \geq 2$, $h \geq 2$.

(1) If there exists $\alpha \in \Phi$ with $k_\alpha(t) < h-1$ then $St_G(t) = 0$ (compare §3.28).

(2) If $k_\alpha(t) \geq h-1$ for all $\alpha \in \Phi$ then there exists a well-defined 'isotropy subgroup' $W(t)$ of the Weyl group W, and $St_G(t) = q^{N(h-1)-l}(q-1)^l \sum_{w \in W(t)} q^{l(w)}$, where we have written $N = |\Phi^+| = n(n-1)/2$, $l = n-1$, and $l(w)$ as in §1.3.

Proof of §5.51

(i) is given in ([16],p.104).

(ii) The calculation is similar to those of §5.34 and §5.4. We shall use the formula of §4.4 :

$$St_G = I(h,h) - I(h,h-1) - I(h-1,h) + I(h-1,h-1).$$

Suppose t has eigenvalues $x,y,z \in R^*$ and suppose $(x-y)\|\pi^k$, $(y-z)\|\pi^l$, $(x-z)\|\pi^m$ with $k \geq l \geq m$; then in fact $l = m$ since $x-z = (x-y) + (y-z)$. By conjugating t by an element of G if necessary we may assume that $t = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$.

Note that $I(h,h-1)(t) = I(h-1,h)(t)$ since t is conjugate to $\theta(t) = \begin{pmatrix} z & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}$.

The modules U_1 of type (h) in V are $\begin{matrix} < e_1+re_2+se_3 > \\ < ae_1+e_2+se_3 > \\ < ae_1+be_2+e_3 > \end{matrix}$ $\begin{matrix} (r,s\in R, \\ a,b\in \pi R). \end{matrix}$

$<e_1+re_2+se_3>$ is fixed by t iff $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}\begin{pmatrix} 1 \\ r \\ s \end{pmatrix} = h\begin{pmatrix} 1 \\ r \\ s \end{pmatrix}$ for

some $h \in R$ iff $h=x,\ hr=yr,\ hs=zs$

iff $(x-y)r=0=(x-z)s$

iff $\pi^{h-k}|r,\ \pi^{h-l}|s$.

Similarly $< ae_1+e_2+se_3 >$ is fixed by t iff $(x-y)a=0=(y-z)s$ iff $\pi^{h-k}|a,\ \pi^{h-l}|s$; and $< ae_1+be_2+e_3 >$ is fixed by t iff $(x-z)a=0=(y-z)b$ iff $\pi^{h-l}|a,\ \pi^{h-l}|b$.

Now let $1\leq i\leq h$; we need only consider the following flags $(U_1 \subset U_2)$ of type (h,i) :

(a) $U_1 = < e_1+re_2+se_3 >$ $(\pi^{h-k}|r,\ \pi^{h-l}|s)$, $U_2 = < U_1, \pi^{h-i}e_2+ce_3 >$ $(\pi^{h-i}|c)$ or $U_2 = < U_1, \pi^{h-i}e_3+de_2 >$ $(\pi^{h-i+1}|d)$.

(b) $U_1 = < ae_1+e_2+se_3 >$ $(\pi^{h-k}|a, \pi^{h-l}|s)$, $U_2 = < U_1, \pi^{h-i}e_1+ce_3 >$ $(\pi^{h-i}|c)$ or $U_2 = < U_1, \pi^{h-i}e_3+de_1 >$ $(\pi^{h-i+1}|d)$.

(c) $U_1 = < ae_1+be_2+e_3 >$ $(\pi^{h-l}|a, \pi^{h-l}|b)$, $U_2 = < U_1, \pi^{h-i}e_1+ce_2 >$ $(\pi^{h-i}|c)$ or $U_2 = < U_1, \pi^{h-i}e_2+de_1 >$ $(\pi^{h-i+1}|d)$.

<u>Case (a)</u>(i): $< U_1, \pi^{h-i}e_2+ce_3 >$ is fixed by t iff there are $h,j \in R$ such that $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}\begin{pmatrix} 0 \\ \pi^{h-i} \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ r \\ s \end{pmatrix} + j\begin{pmatrix} 0 \\ \pi^{h-i} \\ c \end{pmatrix}$

iff $f=0,\ j\pi^{h-i}=y\pi^{h-i},\ jc=zc$

iff $jc=yc$ (since $\pi^{h-i}|c$), $(y-z)c=0$ iff $\pi^{h-l}|c$.

So the number of solutions for c is $\begin{Bmatrix} q^i & \text{if } l \geq i \\ q^l & \text{if } l \leq i \end{Bmatrix}$ and the number of flags of this case fixed by t is $\begin{Bmatrix} q^{k+l+i} & \text{if } l \geq i \\ q^{k+2l} & \text{if } l \leq i. \end{Bmatrix}$

Case (a)(ii): A similar calculation shows that $< U_1, \pi^{h-i}e_3+de_2 >$ is fixed by t iff $\pi^{h-1}|d$; the number of flags of this case fixed by t is $\begin{cases} q^{k+l+i-1} & \text{if } l \geq i-1 \\ q^{k+2l} & \text{if } l \leq i-1 \end{cases}$.

Precisely similar calculations yield the following number of flags fixed by t in the other cases.

Case (b)(i): $\begin{cases} q^{k+l+i} & \text{if } l \geq i \\ q^{k+2l} & \text{if } l \leq i \end{cases}$ if $k \leq h-1$

$\begin{cases} q^{h-1+l+i} & \text{if } l \geq i \\ q^{h-1+2l} & \text{if } l \leq i \end{cases}$ if $k = h$.

(ii): $\begin{cases} q^{k+l+i-1} & \text{if } l \geq i-1 \\ q^{k+2l} & \text{if } l \leq i-1 \end{cases}$ if $k \leq h-1$

$\begin{cases} q^{h-1+l+i-1} & \text{if } l \geq i-1 \\ q^{h-1+2l} & \text{if } l \leq i-1 \end{cases}$ if $k = h$.

Case (c)(i): $\begin{cases} q^{2l+i} & \text{if } k \geq i \\ q^{2l+k} & \text{if } k \leq i \end{cases}$ if $l \leq h-1$

q^{2h-2+i} if $l = h$ (so $k = h$).

(ii): $\begin{cases} q^{2l+i-1} & \text{if } k \geq i-1 \\ q^{2l+k} & \text{if } k \leq i-1 \end{cases}$ if $l \leq h-1$

$q^{2h-2+i-1}$ if $l = h$ (so $k = h$).

A calculation similar to that with modules of type (h) shows that t fixes the following modules of type (h-1).

(a) $< \pi e_1+re_2+se_3 >$ $(\pi^{h-k}|r,\ \pi^{h-1}|s,\ \pi|r,\ \pi|s)$

(b) $< ae_1+\pi e_2+se_3 >$ $(\pi^{h-k}|a,\ \pi^{h-1}|s,\ \pi^2|a,\ \pi|s)$

(c) $< ae_1+be_2+\pi e_3 >$ $(\pi^{h-1}|a,\ \pi^{h-1}|b,\ \pi^2|a,\ \pi^2|b)$.

We only need consider the flags $(W_1 \subset U_2)$ with W_1 one of the above modules of type (h-1) and U_2 as follows.

(a)(i) $< e_1+r'e_2+s'e_3,\ \pi e_2+ce_3 >$ $(\pi|c,\ \pi r'=r,\ \pi s'=s)$

(ii) $< e_1+r'e_2+s'e_3,\ \pi e_3+de_2 >$ $(\pi^2|d,\ \pi r'=r,\ \pi s'=s)$

(b)(i) $< a'e_1+e_2+s'e_3,\ \pi e_1+ce_3 >$ $(\pi|c,\ \pi a'=a,\ \pi s'=s)$

(ii) $< a'e_1+e_2+s'e_3,\ \pi e_3+de_1 >$ $(\pi^2|d,\ \pi a'=a,\ \pi s'=s)$

(c)(i): $< a'e_1+b'e_2+e_3,\ \pi e_1+ce_2 >$ $(\pi|c,\ \pi a'=a,\ \pi b'=b)$

(ii): $< a'e_1+b'e_2+e_3,\ \pi e_2+de_1 >$ $(\pi^2|d,\ \pi a'=a,\ \pi b'=b)$.

Case (a)(i): For U_2 to be fixed by t we require first

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}\begin{pmatrix} 0 \\ \pi \\ c \end{pmatrix} = f\begin{pmatrix} 1 \\ r' \\ s' \end{pmatrix} + j\begin{pmatrix} 0 \\ \pi \\ c \end{pmatrix} \text{ for some } f,j \in R,$$

i.e. $f=0$, $j\pi=y\pi$, $jc=zc$, i.e. $(y-z)c=0$ (since $\pi|c$),

i.e. $\pi^{h-l}|c$;

and also $$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}\begin{pmatrix} 1 \\ r' \\ s' \end{pmatrix} = f\begin{pmatrix} 1 \\ r' \\ s' \end{pmatrix} + j\begin{pmatrix} 0 \\ \pi \\ c \end{pmatrix} \text{ for some } f,j \in R,$$

i.e. $(y-x)r'=j\pi$, $(z-x)s'=jc$.

Note also that for distinct flags we may assume that r' is uniquely determined by r but s' may take q possible values for each s.

If $l \leq h-2$ then $\pi^2|c$ so since $0=r(y-x)=r'\pi(y-x)=j\pi^2$ we see $0=jc=(z-x)s'$ and hence $\pi^{h-l+1}|s$.

If $l = h-1$ then $\pi|c$ and so $(y-x)r'$ determines $\pi^{h-1}s'$.

If $l = h$ there is no extra condition.

Hence the number of flags in this case fixed by t is

$$\begin{cases} q^k.q^{l-1}.q.q^l & \text{if } l \leq h-2,\ k \leq h-1 \\ q^{h-1}.q^{l-1}.q.q^l & \text{if } l \leq h-2,\ k = h \\ q^{h-1}.q^{h-2}.q.q^{h-1} & \text{if } l = h-1 \text{ (so } k \geq h-1) \\ q^{h-1}.q^{h-1}.q.q^{h-1} & \text{if } l = k = h \end{cases}$$

Case (a)(ii): For U_2 to be fixed by t we require $\pi^{h-l}|d$ and $(y-x)r'=jd$, $(z-x)s'=j\pi$ for some $j \in R$. But $\pi^2|d$ so $0=s(z-x)=j\pi^2$ implies $0=(y-x)r'$, i.e. $\pi^{h-k}|r'$. For distinct flags we may assume that s' is uniquely determined by s but r' may take q possible values for each r. The number of fixed flags is $\begin{cases} q^k.q^l.q^l & \text{if } l \leq h-2 \\ q^k.q^{h-1}.q^{h-2} & \text{if } l \geq h-1 \end{cases}$

Proceeding with the other cases in the same way we find the following results.

<u>Case (b)(i)</u>: We require $\pi^{h-1}|c$, $(x-y)a'=j\pi$, $(z-y)s'=jc$. The number of fixed flags is
$$\begin{cases} q^{k+2l} & \text{if } l \le h-2,\ k \le h-2. \\ q^{h-2+2l} & \text{if } l \le h-2,\ k \ge h-1 \\ q^{3h-4} & \text{if } l = h-1,\ (\text{so } k \ge h-1) \\ q^{3h-3} & \text{if } l = h = k \end{cases}$$

<u>Case (b)(ii)</u>: We require $\pi^{h-1}|d$, $(x-y)a'=jd$, $(z-y)s'=j\pi$. The number of fixed flags is
$$\begin{cases} q^{k+2l} & \text{if } l \le h-2,\ k \le h-1 \\ q^{h-1+2l} & \text{if } l \le h-2,\ k = h \\ q^{3h-4} & \text{if } l \ge h-1\ (\text{so } k \ge h-1) \end{cases}$$

<u>Case (c)(i)</u>: We require $\pi^{h-k}|c$, $(x-z)a'=j\pi$, $(y-z)b'=jc$. The number of fixed flags is
$$\begin{cases} q^{k+2l} & \text{if } l \le h-2,\ k \le h-1 \\ q^{h-1+2l} & \text{if } l \le h-2,\ k = h \\ q^{3h-4} & \text{if } l \ge h-1\ (\text{so } k \ge h-1) \end{cases}$$

<u>Case (c)(ii)</u>: We require $\pi^{h-k}|d$, $(x-z)a'=jd$, $(y-z)b'=j\pi$. The number of fixed flags is
$$\begin{cases} q^{k+2l} & \text{if } l \le h-2,\ k \le h-2 \\ q^{h-2+2l} & \text{if } l \le h-2,\ k \ge h-1 \\ q^{3h-5} & \text{if } l \ge h-1\ (\text{so } k \ge h-1) \end{cases}$$

We are now able to compute $St_G(t)$.

<u>Case 1</u>: $l \le k \le h-2$

$I(h,h)(t) = 6q^{k+2l} = I(h,h-1)(t) = I(h-1,h-1)(t)$ so

$St_G(t) = 0$ (see also § 3.28(i)).

<u>Case 2</u>: $l \le h-2$, $k = h-1$

$St_G(t) = q^{h-2+2l}(6q - 2(5q+1) + 4q+2) = 0$

<u>Case 3</u>: $l \le h-2$, $k = h$

$St_G(t) = q^{h-2+2l}(3q^2+3q - 2(2q^2+3q+1) + q^2+3q+2) = 0$

<u>Case 4</u>: $l = h-1 = k$

$St_G(t) = q^{3h-5}(6q^2 - 2(3q^2+3q) + q^2+4q+1) = q^{3h-5}(q-1)^2$

<u>Case 5</u>: $l = h-1$, $k = h$

$St_G(t) = q^{3h-5}(3q^3+3q^2 - 2(q^3+3q^2+2q) + 2q^2+3q+1)$

$\qquad = q^{3h-5}(q^2-1)(q-1)$

<u>Case 6</u>: $l = h = k$

$St_G(t) = q^{3h-5}(q^3-1)(q^2-1)$ (see also § 3.24) ■

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